

FILTERING THE HEEGAARD FLOER CONTACT INVARIANT

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ABSTRACT. We define an invariant of contact structures in dimension 3 from Heegaard Floer homology. This invariant takes values in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$. It is zero for overtwisted contact structures, ∞ for Stein fillable contact structures, non-decreasing under Legendrian surgery, and computable from any supporting open book decomposition. It gives a potentially complete method for determining whether a contact structure is tight, and an obstruction to existence of Stein cobordisms between contact 3-manifolds. We also give a somewhat simpler criterion for tightness, and exhibit examples with vanishing contact invariant in Heegaard Floer homology for which our invariant is finite and non-zero, and which potentially realize any possible value of the invariant.

1. INTRODUCTION

The goal of this article is to define an invariant of closed contact 3-manifolds as a refinement of the contact invariant in Heegaard Floer homology, the *Ozsváth–Szabó contact class* [OS05], and to study some of its properties. Let M be a closed orientable 3-manifold and ξ be a contact structure on M . To define our invariant, we start from an open book decomposition of M supporting ξ and a collection of pairwise disjoint, properly embedded arcs on a page of the open book decomposition. From this data we build a filtered chain complex out of the corresponding Heegaard Floer chain complex whose filtration captures in an algebraic sense the topological complexity of curves counted by the differential. We then consider how far the Ozsváth–Szabó contact class survives in the associated spectral sequence. The result is an invariant of the contact manifold, denoted $\mathfrak{o}(M, \xi)$ and read the *spectral order*, or simply *order*, of (M, ξ) , taking values in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$.

Theorem 1.1. *The contact invariant \mathfrak{o} satisfies the following properties:*

- $\mathfrak{o}(M, \xi) = 0$ if (M, ξ) is overtwisted.
- $\mathfrak{o}(M, \xi) = \infty$ if (M, ξ) is Stein fillable.
- $\mathfrak{o}(M, \xi)$ can be computed from an arbitrary open book decomposition of M supporting the contact structure ξ .

To elaborate on the claim in the last bullet, given an open book decomposition (S, ϕ) of M supporting the contact structure ξ , one can compute the spectral order $\mathfrak{o}(M, \xi)$ using any arc collection with cardinality $6g - 6 + 4B$, subject to some constraints (see Theorem 4.9),

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where g is the genus and B is the number of boundary components of the page S of the open book decomposition.

Moreover, we show that our invariant behaves well under Legendrian surgery, giving a map of partially ordered sets from contact manifolds ordered by Stein cobordisms to the set $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ with the usual ordering. Thus it obstructs Stein cobordisms between contact 3-manifolds.

Theorem 1.2. *The invariant \mathbf{o} is non-decreasing under Legendrian surgery. Equivalently, if (M_-, ξ_-) and (M_+, ξ_+) are respectively the concave and convex ends of a Stein cobordism, then $\mathbf{o}(M_-, \xi_-) \leq \mathbf{o}(M_+, \xi_+)$.*

While the exact calculation of \mathbf{o} in any but particularly simple examples remains a daunting (though straightforward) task, we present in Section 5 a more efficient method to show that $\mathbf{o} > 0$, thus demonstrating tightness of the associated contact structure in cases in which the unrefined contact invariant vanishes and hence gives no information. We illustrate the utility of this method in Section 6. More precisely, we examine an infinite family of contact 3-manifolds $\{(M_{k,m}, \xi_{k,m})\}$, where $k \geq 2$ and $m > k$, with $\hat{c}(\xi_{k,m}) = 0$, which are hence not symplectically fillable. We show that $\mathbf{o}(M_{k,m}, \xi_{k,m}) \leq k$ and, for $k > 2$, $0 < \mathbf{o}(M_{k,m}, \xi_{k,m})$. We expect that $\mathbf{o}(M_{k,m}, \xi_{k,m}) = k$ for all $k \geq 2$ and $m > k$, thus realizing all potential values of the invariant. We plan to return to this matter in the future.

Aside from the properties listed in Theorem 1.1, the contact invariant \mathbf{o} behaves well under connected sums. To be more explicit:

Theorem 1.3. *Let (M_1, ξ_1) and (M_2, ξ_2) be two closed contact 3-manifolds. Then their connected sum satisfies $\mathbf{o}(M_1 \# M_2, \xi_1 \# \xi_2) = \min\{\mathbf{o}(M_1, \xi_1), \mathbf{o}(M_2, \xi_2)\}$.*

This leads to existence of a family of monoids $\mathbf{o}^k(S)$ in the mapping class group $\text{Mod}(S, \partial S)$: $\phi \in \text{Mod}(S, \partial S)$ belongs to $\mathbf{o}^k(S)$ if and only if $\mathbf{o} \geq k$ for the contact 3-manifold specified by the open book decomposition (S, ϕ) (Corollary 4.10).

A consequence of the above is that the connected sum of any contact 3-manifold with finite but non-zero spectral order and a Stein fillable contact 3-manifold is tight but not Stein fillable, which also follows, for example, from combined results of Colin [Col97] and Eliashberg [Eli90]. In particular, one can produce infinite families such as $\{(M_{k,m}, \xi_{k,m})\}$ by taking connected sum of members of this family with a fixed Stein fillable contact 3-manifold.

There is more to be understood about the contact invariant \mathbf{o} and its applications. Some of these are pointed out in Section 7, but those and many others will be deferred for investigation in future work.

It should be noted that our invariant can be thought of as a Heegaard Floer analog of Latschev and Wendl's algebraic torsion [LW11] defined for contact manifolds of arbitrary dimension in the context of symplectic field theory (SFT). An analogous version for contact 3-manifolds was defined by Hutchings in the context of embedded contact homology (ECH) in [LW11, Appendix]. In particular, to a closed oriented 3-manifold M , a nondegenerate contact 1-form λ on M , and a generic almost complex structure J on $\mathbb{R} \times M$ as needed to define the ECH chain complex, Hutchings associates a number in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$. The latter is shown to vanish for overtwisted structures for all choices of λ and J , and can be used

to obstruct exact symplectic cobordisms. However, neither Latschev and Wendl’s algebraic torsion nor Hutchings’s ECH analog is easily computable. Our initial definitions follow the ideas of Hutchings’s construction, ported to the setting of Heegaard Floer homology (see [KMVHMW15] for more on this). While the definition of Latschev and Wendl’s algebraic torsion takes for granted that SFT is well defined, and Hutchings’s ECH analog is not yet known to be independent of the choices of λ and J , \mathfrak{o} is a computable invariant of contact structures on closed 3-manifolds.

Organization. The organization of this article is as follows: Section 2 provides the definitions required throughout the article, in particular of the contact invariant \mathfrak{o} . Section 3 concerns independence of \mathfrak{o} of the several choices made to define it. Section 4 exhibits some properties of \mathfrak{o} , and in doing so proves Theorems 1.1, 1.2, and 1.3. Section 5 presents a sufficient condition for tightness in terms of \mathfrak{o} without resorting to a full computation, while Section 6 presents the aforementioned infinite family of examples $\{(M_{k,m}, \xi_{k,m})\}$ with vanishing Ozsváth–Szabó contact class but with non-zero \mathfrak{o} . Finally, in Section 7, we raise some questions regarding further properties and applications of the contact invariant \mathfrak{o} .

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2. DEFINITIONS

2.1. Background. To set the stage, let M be a closed, connected, and oriented 3-manifold endowed with a co-oriented contact structure ξ . It is understood that the orientation on M is induced by ξ . A celebrated theorem of Giroux states that there is a 1–1 correspondence between contact structures up to isotopy and open book decompositions up to positive stabilization [Gir02]. An abstract open book decomposition of M is a pair (S, ϕ) where S is a compact oriented surface of genus g with B boundary components, called the *page*, and ϕ is an orientation preserving diffeomorphism of S which restricts to identity in a neighborhood of the boundary, called the *monodromy*. The manifold M is diffeomorphic to $S \times [0, 1] / \sim$ where $(p, 1) \sim (\phi(p), 0)$ for any $p \in S$ and $(p, t) \sim (p, t')$ for any $p \in \partial S$ and $t, t' \in [0, 1]$. The open book decomposition is said to support the contact structure ξ if there exists a 1-form λ such that $\xi = \ker(\lambda)$, $\lambda|_{\partial S} > 0$, and $d\lambda|_S > 0$.

Now fix an abstract open book decomposition (S, ϕ) of M supporting ξ and a pairwise disjoint properly embedded collection of arcs $\mathbf{a} = \{a_1, \dots, a_N\}$ on S that contains a basis, that is, a subcollection of arcs cutting S into a polygon. This arc collection together with

the monodromy ϕ defines a Heegaard diagram $(\Sigma, \{\beta_1, \dots, \beta_N\}, \{\alpha_1, \dots, \alpha_N\})$ for $-M$ as in [HKM09b, §3.1]. To be more explicit, let $\mathbf{b} = \{b_1, \dots, b_N\}$ be a collection of arcs on S where b_i is isotopic to a_i via a small isotopy satisfying the following conditions:

- The endpoints of b_i are obtained from the endpoints of a_i by pushing along ∂S in the direction of the boundary orientation,
- a_i intersects b_i transversally at one point, x_i , in the interior of S ,
- Having fixed an orientation of a_i , there is an induced orientation on b_i , and the sign of the oriented intersection $a_i \cap b_i$ is positive. (see Figure 1.)

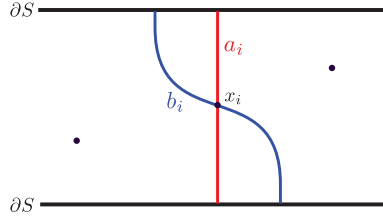


FIGURE 1. The arcs a_i and b_i on the surface S .

Then $\Sigma = S \times \{\frac{1}{2}\} \cup \partial S - S \times \{0\}$, $\alpha_i = a_i \times \{\frac{1}{2}\} \cup a_i \times \{0\}$, and $\beta_i = b_i \times \{\frac{1}{2}\} \cup \phi(b_i) \times \{0\}$. As a parenthetical remark, the Heegaard diagram $(-\Sigma, \{\alpha_1, \dots, \alpha_N\}, \{\beta_1, \dots, \beta_N\})$ also describes the manifold $-M$, and we may sometimes prefer to use this diagram in figures.

With the preceding understood, we recall the definition of the Heegaard Floer chain complex $(\widehat{CF}(\Sigma, \beta, \alpha), \widehat{\partial}_{HF})$. In doing so, we adopt Lipshitz's cylindrical reformulation of Heegaard Floer homology [Lip06]. The definition also requires the choice of basepoints $\mathbf{z} \subset \Sigma \setminus \bigcup_{i \in \{1, \dots, N\}} (\alpha_i \cup \beta_i)$. In the present context, this is done according to the convention in [HKM09b, §3.1]. To be more explicit, place a single basepoint in every connected component of $S \setminus \bigcup_{i \in \{1, \dots, N\}} a_i$ outside the small strips between a_i and b_i (see Figure 1). Following Lipshitz, the chain group $\widehat{CF}(\Sigma, \beta, \alpha)$ is freely generated over $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$ by *I-chord collections* $\vec{\mathbf{x}} := \mathbf{x} \times [0, 1]$ specified by unordered N -tuples of points in Σ of the form $\mathbf{x} = \{x_1, \dots, x_N\}$ where $x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some element σ of the symmetric group S_N . Given a generic almost complex structure J_{HF} on $\Sigma \times [0, 1] \times \mathbb{R}$ satisfying conditions (J1)–(J5) in [Lip06, §1 p. 959], the differential $\widehat{\partial}_{HF}$ on $\widehat{CF}(\Sigma, \beta, \alpha)$ is defined to be the endomorphism of $\widehat{CF}(\Sigma, \beta, \alpha)$ sending a generator $\vec{\mathbf{x}}$ to

$$\sum_{\vec{\mathbf{y}} \in \widehat{\pi}_2(\vec{\mathbf{x}}, \vec{\mathbf{y}})} \sum_{A \in \widehat{\pi}_2(\vec{\mathbf{x}}, \vec{\mathbf{y}}), \text{ind}(A)=1} n(\vec{\mathbf{x}}, \vec{\mathbf{y}}; A) \vec{\mathbf{y}}.$$

Here $\widehat{\pi}_2(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ denotes the set of relative homology classes of continuous maps from a Riemann surface with boundary and boundary punctures into $\Sigma \times [0, 1] \times \mathbb{R}$ such that it maps the boundary of the surface into $\alpha \times \{0\} \times \mathbb{R} \cup \beta \times \{1\} \times \mathbb{R}$, it converges to $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ at its punctures, and it has trivial homological intersection with $\{z\} \times [0, 1] \times \mathbb{R}$. Meanwhile, $\text{ind}(A)$ denotes the index of a class $A \in \widehat{\pi}_2(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ (see [Lip06, Definition 4.4]), and $n(\vec{\mathbf{x}}, \vec{\mathbf{y}}; A)$ is a signed count, modulo \mathbb{R} -translation, of J_{HF} -holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$ satisfying conditions (M0)–(M6) in [Lip06, §1 p. 960] and representing the class A . The latter is

guaranteed to be finite if we choose the monodromy ϕ appropriately in its isotopy class so as to make the multipointed Heegaard diagram $(\Sigma, \beta, \alpha, \mathbf{z})$ *admissible*. A multipointed Heegaard diagram is admissible if every non-trivial periodic domain has both positive and negative coefficients (see [Lip06, Definition 5.1]).

Remark. Even though Lipshitz carried out his construction of a cylindrical reformulation of Heegaard Floer homology in the case $N = 2g + B - 1$ (in other words, the case with one basepoint), the details of his construction and especially the results in [Lip06, §4 and §10] carry over to the multipointed case but for cosmetic changes.

2.2. The filtered chain complex. Next we build a filtered chain complex out of $(\widehat{CF}(\Sigma, \beta, \alpha), \widehat{\partial}_{HF})$. To do this, we adopt Hutchings's recipe in [Hut, §6]. Given a pair of generators $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$, define a function J_+ on $\widehat{\pi}_2(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ by

$$J_+(A) := \mu(\mathcal{D}(A)) - 2e(\mathcal{D}(A)) + |\mathbf{x}| - |\mathbf{y}|^\dagger, \quad (2.1)$$

where $\mathcal{D}(A)$ is the *domain* in the pointed Heegaard diagram $(\Sigma, \beta, \alpha, z)$ representing a class $A \in \widehat{\pi}_2(\vec{\mathbf{x}}, \vec{\mathbf{y}})$, $\mu(\mathcal{D}(A))$ is the *Maslov index* of $\mathcal{D}(A)$ as in the traditional setting of [OS04], $e(\mathcal{D}(A))$ is the *Euler measure* of $\mathcal{D}(A)$ (see [Lip06, §4.1 p. 973] for definition), and $|\cdot|$ denotes the number of cycles in the element of the symmetric group S_N associated to a given generator following the convention described above in Section 2.1. Since the Maslov index and Euler measure are additive under concatenation of domains, so is J_+ . More precisely, for any $A \in \widehat{\pi}_2(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ and $A' \in \widehat{\pi}_2(\vec{\mathbf{y}}, \vec{\mathbf{z}})$, we have

$$J_+(A + A') = J_+(A) + J_+(A').$$

Now suppose that $A \in \widehat{\pi}_2(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ is represented by a J_{HF} -holomorphic curve C_L in $\Sigma \times [0, 1] \times \mathbb{R}$ satisfying conditions (M0)–(M6) in [Lip06, §1]. Then by [Lip06, Proposition 4.2] (cf. [Lip14, Proposition 4.2']),

$$\chi(C_L) = N - n_{\mathbf{x}}(\mathcal{D}(A)) - n_{\mathbf{y}}(\mathcal{D}(A)) + e(\mathcal{D}(A)). \quad (2.2)$$

Here, $n_p(\mathcal{D}(A))$ denotes the *point measure*, namely, the average of the coefficients of $\mathcal{D}(A)$ for the four regions with corners at $p \in \alpha_i \cap \beta_j$. Meanwhile, Lipshitz's formula for the Maslov index of domains [Lip06, Corollary 4.10] (cf. [Lip14, Proposition 4.8']) asserts that

$$\mu(\mathcal{D}(A)) = n_{\mathbf{x}}(\mathcal{D}(A)) + n_{\mathbf{y}}(\mathcal{D}(A)) + e(\mathcal{D}(A)). \quad (2.3)$$

Combining, (2.2) and (2.3), we obtain

$$\mu(\mathcal{D}(A)) - 2e(\mathcal{D}(A)) = -\chi(C_L) + N,$$

and hence (2.1) can be rewritten as

$$J_+(A) = -\chi(C_L) + N + |\mathbf{x}| - |\mathbf{y}|. \quad (2.4)$$

With the preceding understood, consider the smooth compact oriented surface C obtained from the compactification of C_L by attaching 2-dimensional 1-handles along pairs of points in $\alpha_i \times \{0\} \times \mathbb{R} \cap C_L$ and $\beta_i \times \{1\} \times \mathbb{R} \cap C_L$ for each $i = 1, \dots, N$, and then smoothing. Then

[†]The interested reader may refer to [KMVHWMW15] to see how the authors originally came up with this formula.

$\chi(C) = \chi(C_L) - N$, and $|\mathbf{x}|$ (resp. $|\mathbf{y}|$) is equal to the number of boundary components of C arising from the I -chord $\vec{\mathbf{x}}$ (resp. $\vec{\mathbf{y}}$). Hence, we can further rewrite (2.4) as

$$J_+(A) = \sum_{C_j \subset C} (2g_j - 2 + 2|\mathbf{x}_j|), \quad (2.5)$$

where each C_j denotes a connected component of C , g_j denotes the genus of C_j , and each $\mathbf{x}_j \subset \mathbf{x}$ denotes the maximal subcollection of points in \mathbf{x} such that $\mathbf{x}_j \times [0, 1]$ lies on the boundary of the component C_j . Note that each connected component of C has non-empty intersections with the I -chord collections specified by \mathbf{x} and \mathbf{y} since each connected component of C_L has non-empty negative and positive ends. Therefore, it follows from (2.5) that $2 \mid J_+(A)$, and $J_+(A) \geq 0$.

Remark. If there exists an embedded J_{HF} -holomorphic curve C_L representing the class A , then the Maslov index of $\mathcal{D}(A)$ agrees with the Fredholm index of C_L . For Maslov index-1 domains, we prefer to use the following equivalent formula:

$$J_+(A) = 2[n_{\mathbf{x}}(\mathcal{D}(A)) + n_{\mathbf{y}}(\mathcal{D}(A))] - 1 + |\mathbf{x}| - |\mathbf{y}|. \quad (2.6)$$

2.3. An analog of algebraic torsion. Following verbatim [LW11, Appendix], we decompose the Heegaard Floer differential as

$$\widehat{\partial}_{HF} = \partial_0 + \partial_1 + \cdots + \partial_\ell + \cdots,$$

where ∂_ℓ counts J_{HF} -holomorphic curves with $J_+ = 2\ell$ and having empty intersection with $\{z\} \times [0, 1] \times \mathbb{R}$. Since J_+ is additive under gluing of J -holomorphic curves, the above decomposition induces a spectral sequence with pages

$$E^k(S, \phi, \mathbf{a}; J_{HF}) = H_*(E^{k-1}(S, \phi, \mathbf{a}; J_{HF}), d_{k-1}).$$

To be more explicit, consider the \mathbb{Z} -graded direct sum

$$\widehat{\mathcal{CF}}(S, \phi, \mathbf{a}) := \widehat{CF}(\Sigma, \beta, \alpha) \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}]$$

endowed with the endomorphism $\widehat{\partial}$ defined by

$$\widehat{\partial}(\sum_{i \in \mathbb{Z}} c_i t^i) := \sum_{i \in \mathbb{Z}} (\sum_{\ell \in \mathbb{Z}} (\partial_\ell c_i) t^{i-\ell}).$$

Here $c_i \neq 0$ for only finitely many $i \in \mathbb{Z}$. Note that the additivity property of J_+ implies that

$$\sum_{i+j=\ell} \partial_i \circ \partial_j = 0,$$

for any $\ell \geq 0$; hence, $\widehat{\partial} \circ \widehat{\partial} = 0$ making $(\widehat{\mathcal{CF}}(S, \phi, \mathbf{a}), \widehat{\partial})$ into a filtered chain complex where the p^{th} filtration level

$$\mathcal{F}^p(S, \phi, \mathbf{a}) = \{\sum_{i \leq p} c_i t^i \mid c_i \in \widehat{CF}(\Sigma, \beta, \alpha)\}.$$

Then $(E^k(S, \phi, \mathbf{a}; J_{HF}), d_k)$ is the spectral sequence associated to this filtered chain complex where d_k is the restriction of $\widehat{\partial}$ to $E^k(S, \phi, \mathbf{a}; J_{HF})$. To be more explicit, let A_p^r denote the subcomplex defined by

$$A_p^r = \{c \in \mathcal{F}^p(S, \phi, \mathbf{a}) \mid \widehat{\partial} c \in \mathcal{F}^{p-r}(S, \phi, \mathbf{a})\}.$$

Then,

$$E_p^k(S, \phi, \mathbf{a}; J_{HF}) = \frac{A_p^k}{\widehat{\partial} A_{p+k-1}^{k-1} + A_{p-1}^{k-1}}.$$

A simple computation shows that $E_p^k(S, \phi, \mathbf{a}; J_{HF})$ is isomorphic to

$$\frac{Z^k(S, \phi, \mathbf{a}; J_{HF})}{\mathcal{B}^k(S, \phi, \mathbf{a}; J_{HF})},$$

where

$$\begin{aligned} Z^k(S, \phi, \mathbf{a}; J_{HF}) &:= \{c \in \widehat{CF}(\Sigma, \beta, \alpha) \mid \exists c_i \in \widehat{CF}(\Sigma, \beta, \alpha) \text{ for } i = 1, \dots, k-1 \\ &\quad \text{s.t. } \partial_0 c = 0 \text{ and } \partial_j c = \sum_{i=0}^{j-1} \partial_i c_{j-i} \text{ for } 0 < j < k\}, \\ \mathcal{B}^k(S, \phi, \mathbf{a}; J_{HF}) &:= \left\{ \sum_{i=0}^{k-1} \partial_i b_i \mid b_i \in \widehat{CF}(\Sigma, \beta, \alpha) \text{ and } \sum_{i=\ell=j} \partial_\ell b_i = 0 \text{ for } 0 < j < k \right\}, \end{aligned}$$

and since $\mathcal{F}^p(S, \phi, \mathbf{a}) \cong \mathcal{F}^{p-1}(S, \phi, \mathbf{a})$, the above quotient completely determines the pages of the spectral sequence.

By [HKM09b, Theorem 3.1], the Heegaard Floer generator specified by the set of distinguished points $\mathbf{x}_\xi = \{x_1, \dots, x_N\}$ indicated in Figure 1 represents the Ozsváth–Szabó contact class $\widehat{c}(\xi) \in \widehat{HF}(-M)$, and it satisfies $\partial_i \vec{\mathbf{x}}_\xi = 0$ for all $i \geq 0$. This is because there is no Fredholm index-1 J_{HF} -holomorphic curve in $\Sigma \times [0, 1] \times \mathbb{R}$ satisfying conditions (M0)–(M6) in [Lip06, §1] with $\vec{\mathbf{x}}_\xi$ at its negative punctures and having empty intersection with $\{z\} \times [0, 1] \times \mathbb{R}$. Hence, $\vec{\mathbf{x}}_\xi$ represents a cycle in $E^k(S, \phi, \mathbf{a}; J_{HF})$ for all $k \geq 1$, and it is natural to ask how far into the spectral sequence does $\vec{\mathbf{x}}_\xi$ survive.

Definition 2.1. Define $o(S, \phi, \mathbf{a}; J_{HF})$ to be the smallest non-negative integer k such that the generator $\vec{\mathbf{x}}_\xi$ represents the trivial class in $E^{k+1}(S, \phi, \mathbf{a}; J_{HF})$.

Ideally, one would like to show that $o(S, \phi, \mathbf{a}; J_{HF})$ does not depend on choices of (S, ϕ, \mathbf{a}) and J_{HF} . This is not true in general. For example, consider the closed contact 3-manifold where the contact structure is supported by the open book decomposition (S, ϕ) depicted in Figure 2a. Using the basis of arcs \mathbf{a} shown in Figure 2a, and a generic *split* almost complex structure J_{HF} , we observe that the shaded domain \mathcal{D} in Figure 2b is sufficient for the vanishing of the Ozsváth–Szabó contact class. A simple computation shows that $J_+(\mathcal{D}) = 2$. Therefore, $\vec{\mathbf{x}}_\xi$ represents the trivial class in $E^2(S, \phi, \mathbf{a}; J_{HF})$, and $o(S, \phi, \mathbf{a}; J_{HF}) \leq 1$. Furthermore, using the symmetry of the open book decomposition and the choice of the arc basis, one can argue as in Section 6 that $o(S, \phi, \mathbf{a}; J_{HF}) = 1$. However, the contact structure supported by the open book decomposition (S, ϕ) is overtwisted, which can be seen after a sequence of positive stabilizations to reveal the overtwisted disk (see [Wan15a] for an explicit algorithm). Then there exists another open book decomposition (S', ϕ') and a basis of arcs \mathbf{a}' on S' for which $o(S', \phi', \mathbf{a}'; J'_{HF}) = 0$ using a generic split almost complex structure J'_{HF} (see proof of Theorem 4.1). As a result, o is not independent of these choices. That is why we introduce the next definition.

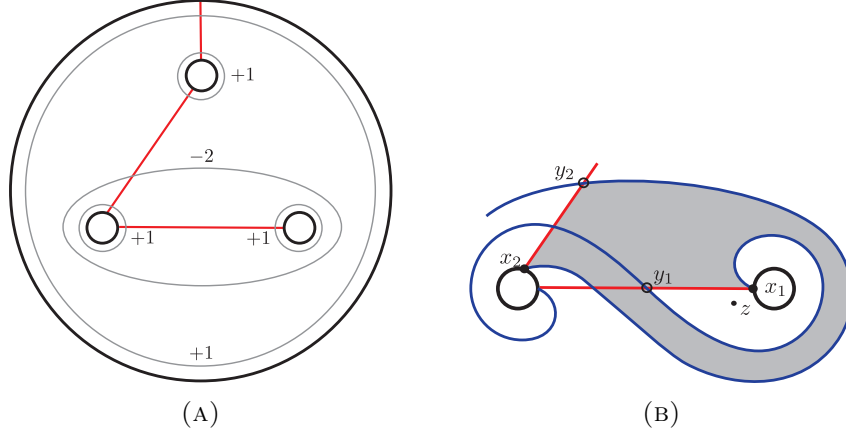


FIGURE 2. On the left is an open book decomposition (S, ϕ) supporting an overtwisted contact structure and a basis of arcs depicted in red. On the right is a Maslov index-1 holomorphic domain with $J_+ = 2$ in the $S \times \{0\}$ half of the Heegaard diagram $(-\Sigma, \alpha, \beta)$.

Definition 2.2. Let (M, ξ) be a closed contact 3-manifold. Then define the *spectral order*

$$\mathbf{o}(M, \xi) := \min\{o(S, \phi, \mathbf{a}; J_{HF})\},$$

where the minimum is taken over all data $(S, \phi, \mathbf{a}; J_{HF})$ such that (S, ϕ) is an open book decomposition of M supporting ξ , \mathbf{a} is pairwise disjoint properly embedded collection of arcs on S that contains a basis, and J_{HF} is a generic almost complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$ satisfying conditions (J1)–(J5) in [Lip06, §1].

It follows immediately that Definition 2.2 yields an invariant of contact structures. However, proving Theorem 1.1 requires introducing a more tractable version of \mathbf{o} . In this regard, we restrict attention to *complete sets of arcs*, namely, maximal collections of pairwise disjoint properly embedded non-isotopic arcs.

Definition 2.3. Let (M, ξ) be a closed contact 3-manifold. Fix an open book decomposition $\mathcal{B} = (S, \phi)$ of M supporting ξ . Then define

$$\mathfrak{o}(\mathcal{B}) := \min\{o(S, \phi, \mathbf{a}; J_{HF})\},$$

where the minimum is taken over all choices of \mathbf{a} and J_{HF} such that \mathbf{a} is a complete set of arcs on S and J_{HF} is a generic almost complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$ satisfying conditions (J1)–(J5) in [Lip06, §1]. Also define

$$\mathbf{o}^c(M, \xi) := \min\{\mathfrak{o}(\mathcal{B})\},$$

where the minimum is taken over all open book decompositions \mathcal{B} of M supporting ξ .

Once again, Definition 2.3 yields an invariant of contact structures. We show in Section 3 that \mathbf{o} is independent of choices of generic almost complex structure J_{HF} on $\Sigma \times [0, 1] \times \mathbb{R}$, and monodromy ϕ on S up to isotopy relative to ∂S . Moreover, it assumes the same value for any choice of complete set of arcs. Therefore, \mathfrak{o} defines an invariant of open book

decompositions. This, along with the fact that $\mathbf{o}^c(M, \xi) = \mathbf{o}(M, \xi)$ for any closed contact 3-manifold (M, ξ) (see Proposition 4.5), allows us to complete the proof of Theorem 1.1 in Section 4.

3. INDEPENDENCE OF CHOICES

This section investigates dependence of $o(S, \phi, \mathbf{a}; J_{HF})$ on choices of open book decomposition (S, ϕ) of M supporting ξ , generic almost complex structure J_{HF} on $\Sigma \times [0, 1] \times \mathbb{R}$, where $\Sigma = S \times \{\frac{1}{2}\} \cup_{\partial S} -S \times \{0\}$, and complete set of arcs on S . We start with *a priori* dependence of o on a choice of generic almost complex structure.

Proposition 3.1. *Fix an open book decomposition (S, ϕ) of M supporting ξ and a pairwise disjoint properly embedded collection of arcs \mathbf{a} on S that contains a basis. Suppose that (S, ϕ, \mathbf{a}) yields an admissible Heegaard diagram, and let J_{HF}^0 and J_{HF}^1 be two generic almost complex structures on $\Sigma \times [0, 1] \times \mathbb{R}$ satisfying conditions (J1)–(J5) in [Lip06, §1]. Then $o(S, \phi, \mathbf{a}; J_{HF}^0) = o(S, \phi, \mathbf{a}; J_{HF}^1)$.*

Proof. There exists a smooth 1-parameter family of \mathbb{R} -invariant almost complex structures $\{J_{HF}^s\}_{s \in \mathbb{R}}$ on $\Sigma \times [0, 1] \times \mathbb{R}$ that agrees with J_{HF}^0 if $s < \epsilon$ and with J_{HF}^1 if $s > 1 - \epsilon$ for some $\epsilon \ll 1$. As is explained in [Lip06, §9], this family of almost complex structures can be chosen to satisfy conditions (J1), (J2), and (J4) in [Lip06, §1] when considered as a non- \mathbb{R} -invariant almost complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$. Furthermore, this almost complex structure guarantees transversality for pseudo-holomorphic curves with prescribed boundary conditions. It is used in [Lip06, §9] to define a chain map

$$\Phi : (\widehat{CF}(\Sigma, \beta, \alpha), \widehat{\partial}_{HF}^0) \rightarrow (\widehat{CF}(\Sigma, \beta, \alpha), \widehat{\partial}_{HF}^1)$$

via a signed count of J_{HF}^s -holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$ satisfying conditions (M0)–(M6) in [Lip06, §1] and representing relative homology classes $A \in \widehat{\pi}_2(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ with $\text{ind}(A) = 0$. If J_{HF}^s is generic, then the moduli space of such J_{HF}^s -holomorphic curves representing a class $A \in \widehat{\pi}_2(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ with $\text{ind}(A) = 0$, respectively $\text{ind}(A) = 1$, is a smooth orientable 0-dimensional, respectively 1-dimensional, manifold whose compactification in the 1-dimensional case is obtained by adding on pseudo-holomorphic buildings of height 2 in which one level is J_{HF}^s -holomorphic and the other is either J_{HF}^0 -holomorphic or J_{HF}^1 -holomorphic as the case may be. The topology of the curves in each component of these moduli spaces is fixed.

Now we define an integer valued function on moduli spaces of J_{HF}^s -holomorphic curves in $\Sigma \times [0, 1] \times \mathbb{R}$ with $\text{ind} \leq 1$ satisfying conditions (M0)–(M6) in [Lip06, §1]. If C_L is such a curve representing a class in $\widehat{\pi}_2(\vec{\mathbf{x}}, \vec{\mathbf{y}})$, then define

$$J_+(C_L) := -\chi(C_L) + N + |\mathbf{x}| - |\mathbf{y}|. \quad (3.1)$$

Note that (3.1) is additive in the sense that if a pseudo-holomorphic building of height 2 consists of a J_{HF}^0 -holomorphic curve C_L^1 with $\text{ind} = 1$ representing a class in $\widehat{\pi}_2(\vec{\mathbf{x}}, \vec{\mathbf{x}}')$ and a J_{HF}^s -holomorphic curve C_L^0 with $\text{ind} = 0$ representing a class in $\widehat{\pi}_2(\vec{\mathbf{x}}', \vec{\mathbf{y}})$, then the J_{HF}^s -holomorphic curve C_L obtained from these by gluing (see [Lip06, Appendix A]) satisfies

$$J_+(C_L) = J_+(C_L^1) + J_+(C_L^0), \quad (3.2)$$

since $\chi(C_L) = \chi(C_L^1) + \chi(C_L^0) - N$. The same holds for a pseudo-holomorphic building of height 2 consisting of a J_{HF}^s -holomorphic curve C_L^0 with $ind = 0$ representing a class in $\hat{\pi}_2(\vec{x}, \vec{y}')$ and a J_{HF}^1 -holomorphic curve C_L^1 with $ind = 1$ representing a class in $\hat{\pi}_2(\vec{y}', \vec{y})$. Note also that (3.1) coincides with (2.4), which allows us to deduce similarly that $J_+(C_L)$ is a non-negative even integer. Hence, we may decompose Φ as

$$\Phi = \Phi^0 + \Phi^1 + \cdots + \Phi^\ell + \cdots,$$

where Φ^ℓ counts J_{HF}^s -holomorphic curves with $J_+ = 2\ell$. Since Φ is a chain map, and J_+ is additive under gluing, it follows that

$$\sum_{i+j=\ell} (\Phi^i \circ \partial_j^0 - \partial_i^1 \circ \Phi^j) = 0.$$

This identity implies that there is a filtered chain map $\hat{\Phi}$ from $(\widehat{\mathcal{CF}}(S, \phi, \mathbf{a}), \hat{\partial}^0)$ to $(\widehat{\mathcal{CF}}(S, \phi, \mathbf{a}), \hat{\partial}^1)$ defined by

$$\hat{\Phi}(\sum_{i \in \mathbb{Z}} c_i t^i) := \sum_{i \in \mathbb{Z}} \left(\sum_{\ell \in \mathbb{Z}} (\Phi^\ell c_i) t^{i-\ell} \right),$$

hence a morphism of spectral sequences from $E^*(S, \phi, \mathbf{a}; J_{HF}^0)$ to $E^*(S, \phi, \mathbf{a}; J_{HF}^1)$. Moreover, $\Phi(\vec{x}_\xi) = \vec{x}_\xi$ since the only J_{HF}^s -holomorphic curve with negative ends at \vec{x}_ξ satisfying conditions (M0)–(M6) in [Lip06, §1] is $\vec{x}_\xi \times \mathbb{R}$. Therefore, we have $o(S, \phi, \mathbf{a}; J_{HF}^0) \geq o(S, \phi, \mathbf{a}; J_{HF}^1)$. On the other hand, we may also consider the chain map induced by the smooth 1-parameter family of almost complex structures $\{J_{HF}^{1-s}\}_{s \in \mathbb{R}}$. Likewise, we obtain $o(S, \phi, \mathbf{a}; J_{HF}^0) \leq o(S, \phi, \mathbf{a}; J_{HF}^1)$. As a result, $o(S, \phi, \mathbf{a}; J_{HF}^0) = o(S, \phi, \mathbf{a}; J_{HF}^1)$. \square

3.1. Isotopy independence. What with Proposition 3.1, we may drop a choice of generic almost complex structure from the notation and simply write $o(S, \phi, \mathbf{a})$. We proceed to discuss the dependence of o on the monodromy. In this regard, let ϕ and ϕ' be two orientation-preserving diffeomorphisms of S that restrict to identity in a neighborhood of ∂S . Suppose that ϕ is isotopic to ϕ' , and fix an isotopy $\{\phi_t\}_{t \in [0,1]}$ relative to ∂S such that $\phi_0 = \phi$ and $\phi_1 = \phi'$. Given a pairwise disjoint properly embedded collection of arcs \mathbf{a} on S that contains a basis, the isotopy $\{\phi_t\}_{t \in [0,1]}$ yields an isotopy of arcs $\{\phi_t(\mathbf{b})\}_{t \in [0,1]}$, where \mathbf{b} is the collection of arcs as in Section 2.1. Of interest to us are two kinds of isotopies:

- (1) For any $t \in [0, 1]$, \mathbf{a} intersects $\phi_t(\mathbf{b})$ transversally in the interior of S .
- (2) The isotopy creates/annihilates a pair of transverse intersections between \mathbf{a} and $\phi(\mathbf{b})$.

Following [Lip06], we refer to such isotopies as *basic isotopies*. In general, a pointed isotopy between two multipointed Heegaard diagrams, namely, an isotopy supported in the complement of the basepoints, is called *admissible* if each intermediate multipointed Heegaard diagram is admissible. Any two admissible multipointed Heegaard diagrams that are pointed isotopic are in fact isotopic through a sequence of admissible basic isotopies (see [Lip06, Proposition 5.6]). Note that isotopies of the monodromy of an open book decomposition yield pointed isotopies of the corresponding multipointed Heegaard diagram. Therefore, it suffices to investigate the behavior of o under admissible basic isotopies of the monodromy.

Proposition 3.2. *Let (S, ϕ) be an open book decomposition and \mathbf{a} be a pairwise disjoint properly embedded collection of arcs \mathbf{a} on S that contains a basis. Suppose that (S, ϕ, \mathbf{a}) yields an admissible multipointed Heegaard diagram, and that ϕ' is isotopic to ϕ via an admissible basic isotopy. Then $o(S, \phi', \mathbf{a}) = o(S, \phi, \mathbf{a})$.*

Proof. As is explained in [Lip06, Chapter 9] (cf. [OS04, §7.3]), basic isotopies of the first kind above are equivalent to deformations of the complex structure on Σ . With this understood, o is unchanged under isotopies of this sort by Proposition 3.1. As for basic isotopies of the second kind above, we consider the chain maps induced by the multipointed Heegaard triple diagram $(\Sigma, \beta', \beta, \alpha, \mathbf{z})$ where $\beta' = \{\beta'_1, \dots, \beta'_N\}$ is such that each β'_i is obtained from a small Hamiltonian isotopy of $a_i \cup \phi'(b_i)$ so that it intersects β_i transversally in exactly two points near the point x_i as shown in Figure 3, while it is disjoint from β_j for $j \neq i$. As a result, the Heegaard diagram (Σ, β', β) represents the manifold $\#_N S^1 \times S^2$, we may assume that β' is sufficiently close to β so that the signed area of the region between them is zero with respect to an area form on Σ which delivers the admissibility criteria for the multipointed Heegaard diagram $(\Sigma, \beta', \beta, \mathbf{z})$ as stated in [Lip06, Lemma 5.3]. Consequently, the multipointed Heegaard triple diagram $(\Sigma, \beta', \beta, \alpha, \mathbf{z})$ is also admissible by [Lip06, Lemma 10.14].

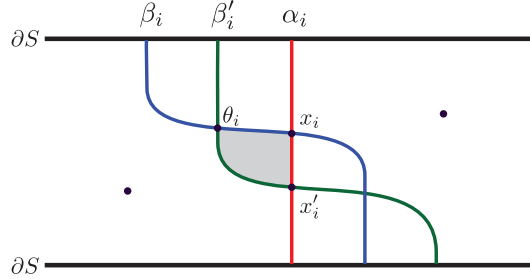


FIGURE 3. Part of the restriction of the multipointed Heegaard triple diagram $(\Sigma, \beta', \beta, \alpha, \mathbf{z})$ to $S \times \{\frac{1}{2}\} \subset \Sigma$.

The Heegaard triple diagram $(\Sigma, \beta', \beta, \alpha)$ describes a cobordism with one outgoing boundary component and two incoming boundary components, one of which is diffeomorphic to the manifold $\#_N S^1 \times S^2$. To be more specific, this cobordism is diffeomorphic to the complement of a tubular neighborhood of a bouquet of N embedded circles in the product cobordism $[0, 1] \times M$. It follows that, there is a unique Spin^c structure \mathfrak{t}_ξ on this cobordism which restricts to the trivial Spin^c structure \mathfrak{s}_0 on $\#_N S^1 \times S^2$ and to \mathfrak{s}_ξ on M .

With the preceding understood, there exists a chain map

$$\widehat{f}_{\beta', \beta, \alpha; \mathfrak{t}_\xi} : \widehat{CF}(\Sigma, \beta', \beta, \mathfrak{s}_0) \otimes_{\mathbb{F}} \widehat{CF}(\Sigma, \beta, \alpha, \mathfrak{s}_\xi) \rightarrow \widehat{CF}(\Sigma, \beta', \alpha, \mathfrak{s}_\xi),$$

defined by counting embedded Fredholm index-0 pseudo-holomorphic curves in $\Sigma \times T$ subject to appropriate boundary conditions. Here T denotes a disk with three marked points on its boundary and $\Sigma \times T$ is equipped with an almost complex structure satisfying conditions (J'1)–(J'4) in [Lip06, §10.2 p. 1018].

No matter the almost complex structure, the differential on $\widehat{CF}(\Sigma, \beta', \beta, \mathfrak{s}_o)$ vanishes identically. Therefore, restricting to the subcomplex $\mathbb{F} \cdot \vec{\theta} \otimes_{\mathbb{F}} \widehat{CF}(\Sigma, \beta, \alpha, \mathfrak{s}_\xi)$, where $\theta = \{\theta_1, \dots, \theta_N\}$ and $\vec{\theta}$ is the top degree generator of $\widehat{CF}(\Sigma, \beta', \beta, \mathfrak{s}_o)$, results in a chain map

$$\hat{f}_{\beta', \beta, \alpha; \mathfrak{t}_\xi}(\vec{\theta} \otimes \cdot) : \widehat{CF}(\Sigma, \beta, \alpha, \mathfrak{s}_\xi) \rightarrow \widehat{CF}(\Sigma, \beta', \alpha, \mathfrak{s}_\xi).$$

The latter induces an isomorphism of homologies by [Lip06, Proposition 11.4] (cf. [OS04, Proposition 9.8]). In what follows, we work with a generic split complex structure on $\Sigma \times T$. We are allowed to do so since transversality of moduli spaces as defined by such almost complex structures can be guaranteed by slight perturbation of the α , β , and β' curves. To be more precise, we may invoke the technique of [Oh96]. This is because any class A in $\hat{\pi}_2(\vec{\theta}, \cdot, \cdot)$ satisfies the boundary injectivity criterion in the sense of [Lip06]. By way of a reminder, a class A in $\hat{\pi}_2(\vec{\theta}, \cdot, \cdot)$ is said to satisfy the boundary injectivity criterion if any pseudo-holomorphic curve u for some split complex structure on $\Sigma \times T$ representing the class A has $\pi_\Sigma \circ u$ somewhere injective in its boundary. This criterion is guaranteed as long as the domain representing the class has two adjacent regions, one with multiplicity zero and the other with multiplicity one. Note that this is the case for any class in $\hat{\pi}_2(\vec{\theta}, \cdot, \cdot)$ due to the placement of the basepoints in that basepoints appear on both sides of every α , β , and β' curve.

Next we show that the chain map $\hat{f}_{\beta', \beta, \alpha; \mathfrak{t}_\xi}(\vec{\theta} \otimes \cdot)$ induces a morphism of spectral sequences from $E^*(S, \phi, \mathbf{a}; J_{HF})$ to $E^*(S, \phi', \mathbf{a}; J'_{HF})$. First, define an analog of the formula (2.1) for the cobordism described by the Heegaard triple diagram $(\Sigma, \beta', \beta, \alpha)$ via

$$J_+(A) = \frac{N}{2} + \mu(\mathcal{D}(A)) - 2e(A) + |\mathbf{x}| - |\mathbf{y}|, \quad (3.3)$$

where $A \in \hat{\pi}_2(\vec{\theta}, \vec{\mathbf{x}}, \vec{\mathbf{y}})$, $\mu(\mathcal{D}(A))$ denotes the Maslov index of the domain $\mathcal{D}(A)$ associated to A , which is the expected dimension of the moduli space of pseudo-holomorphic curves representing the class A , and $e(A)$ is the Euler measure of the domain associated to the class A . If A can be represented by an embedded Fredholm index-0 pseudo-holomorphic curve C_L , then the formula (3.3) becomes

$$\begin{aligned} J_+(A) &= \frac{N}{2} - 2e(A) + |\mathbf{x}| - |\mathbf{y}| \\ &= \underbrace{-\chi(C_L) + N}_{\text{by [Lip06, §10.2]}} + |\mathbf{x}| - |\mathbf{y}|. \end{aligned}$$

It follows from this formula that $J_+(A) = 2\ell$ for some $\ell \geq 0$. To see this, consider the smooth compact oriented surface C obtained from the compactification of C_L by first adding 2-dimensional 1-handles, one for each pair (β'_i, β_i) and one for each pair (β'_i, α_i) , and then capping off the boundary components of the resulting surface containing the I -chord collection $\vec{\theta}$. Note that $\chi(C) = \chi(C_L) - N$, and $|\mathbf{x}|$ (resp. $|\mathbf{y}|$) is equal to the number of boundary components of C arising from the I -chord collection $\vec{\mathbf{x}}$ (resp. $\vec{\mathbf{y}}$). The claim then follows in exactly the same way as in §2. Consequently, we can decompose the chain map $\hat{f}_{\beta', \beta, \alpha; \mathfrak{t}_\xi}(\vec{\theta} \otimes \cdot)$ as

$$\hat{f}_{\beta', \beta, \alpha; \mathfrak{t}_\xi}(\vec{\theta} \otimes \cdot) = f^0 + f^1 + \dots + f^\ell + \dots,$$

where f^ℓ counts embedded Fredholm index-0 pseudo-holomorphic curves with $J_+ = 2\ell$. Since the Maslov index and the Euler measure are additive under concatenation, it follows using the formulae (2.1) and (3.3) that J_+ is also additive. Therefore, we have

$$\sum_{i+j=\ell} (f^i \circ \partial_j - \partial'_i \circ f^j) = 0 \quad (3.4)$$

since $\hat{f}_{\beta', \beta, \alpha; t_\xi}$ is a chain map and the J_+ -filtered differential on $\widehat{CF}(\Sigma, \beta', \beta, \mathfrak{s}_o)$ is identically zero. The latter is due to the fact that $\widehat{CF}(\Sigma, \beta', \beta, \mathfrak{s}_o)$ is isomorphic to $(\mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)})^{\otimes N}$ where $\mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)}$ is a graded module over \mathbb{F} with vanishing differential and the domains corresponding to the pseudo-holomorphic curves that contribute to the differential of the generator, $\theta_i \times [0, 1]$, of $\mathbb{F}_{(1)}$ are both bigons, which have $J_+ = 0$. In short, the restriction of the differential on $\widehat{CF}(\Sigma, \beta', \beta, \mathfrak{s}_o) \otimes_{\mathbb{F}} \widehat{CF}(\Sigma, \beta, \alpha, \mathfrak{s}_\xi)$ to the subcomplex $\mathbb{F} \cdot \vec{\theta} \otimes_{\mathbb{F}} \widehat{CF}(\Sigma, \beta, \alpha, \mathfrak{s}_\xi)$ is J_+ -filtered.

The identity (3.4) implies that there is a filtered chain map from $(\widehat{CF}(S, \phi, \mathbf{a}), \hat{\partial})$ to $(\widehat{CF}(S, \phi', \mathbf{a}), \hat{\partial}')$ as before, hence a morphism of spectral sequences from $E^*(S, \phi, \mathbf{a}; J_{HF})$ to $E^*(S, \phi', \mathbf{a}; J'_{HF})$. In addition,

$$\hat{f}_{\beta', \beta, \alpha; t_\xi}(\vec{\theta} \otimes \vec{x}_\xi) = \vec{x}'_\xi$$

since the shaded triangles in Figure 3 constitute the only holomorphic domain that contributes to this chain map due to the placement of the basepoints, and it is represented by a unique pseudo-holomorphic curve by the Riemann Mapping Theorem. Hence, $o(S, \phi, \mathbf{a}; J_{HF}) \geq o(S, \phi', \mathbf{a}; J'_{HF})$. Likewise, the isotopy from ϕ' to ϕ yields $o(S, \phi, \mathbf{a}; J_{HF}) \leq o(S, \phi', \mathbf{a}; J'_{HF})$. As a result, $o(S, \phi, \mathbf{a}; J_{HF}) = o(S, \phi', \mathbf{a}; J'_{HF})$. \square

Remark. Sarkar–Wang [SW10] and Plamenevskaya [Pla07] proved that the Heegaard diagram resulting from an arbitrary choice of (S, ϕ, \mathbf{a}) , where \mathbf{a} contains a basis, can be made *nice* by choosing ϕ appropriately in its isotopy class. On a nice Heegaard diagram, every Maslov index-1 holomorphic domain is represented by an *empty* embedded bigon or an *empty* embedded square [SW10, Theorem 3.3]. It is easy to see from (2.1) that such domains have either $J_+ = 0$ or $J_+ = 2$. This observation indicates that there should be a combinatorial description of o (see Part 2 of Section 6 for an example computation), hence a combinatorial description of \mathbf{o} via Theorem 4.9.

3.2. Independence of complete arc collections. The next proposition in this section asserts that o does not depend on a choice of complete set of arcs. As we shall see in Section 4, this gives us quite a bit of flexibility in our arguments that lead to the proofs of our main theorems.

Proposition 3.3. *Let (S, ϕ) be an open book decomposition of M supporting a contact structure ξ . Then $o(S, \phi, \mathbf{a})$ does not depend on a choice of complete set of arcs \mathbf{a} on S .*

Proof. To start, given a smooth compact oriented surface with boundary, one can pass from any complete set of arcs to any other complete set of arcs through a sequence of *Pachner moves* [Pac91], namely, bistellar flips in the resulting triangulation of the surface obtained

by collapsing each boundary component to a point. Therefore, it suffices to prove that o does not change under Pachner moves.

An open book decomposition (S, ϕ) of M supporting ξ and a complete set of arcs on S determines a multipointed Heegaard diagram $(\Sigma, \beta, \alpha, \mathbf{z})$ for $-M$. After a Pachner move, the resulting complete set of arcs on S specifies a new multipointed Heegaard diagram $(\Sigma, \beta', \alpha', \mathbf{z})$ for $-M$. The two Heegaard diagrams are related by a sequence of handle-slides. Therefore, there is a chain map from $\widehat{CF}(\Sigma, \beta, \alpha)$ to $\widehat{CF}(\Sigma, \beta', \alpha')$ induced by handle-slide cobordisms. We would like to show that this chain map induces a filtered chain map between the corresponding filtered chain complexes, hence a morphism between the associated spectral sequences. To do so, we consider this chain map as the composition of two chain maps induced by the cobordisms described by the multipointed Heegaard triple diagrams $(\Sigma, \beta, \alpha, \alpha', \mathbf{z})$ and $(\Sigma, \beta', \beta, \alpha', \mathbf{z})$. Our goal is to show that each of these chain maps induce filtered chain maps, and the composite map sends the contact generator in $\widehat{CF}(\Sigma, \beta, \alpha)$ to the contact generator in $\widehat{CF}(\Sigma, \beta', \alpha')$.

Let $\mathbf{a} = \{a_1, a_2, \dots, a_N\}$ be a complete set of arcs on S and $\mathbf{a}' = \{a'_1, a'_2, \dots, a'_N\}$ be the complete set of arcs on S after a Pachner move. By Proposition 3.2, we may choose α'_i to be a small Hamiltonian translate of α_i so that it intersects α_i transversally in exactly two distinct points near the unique intersection point of a_i and b_i if $i \neq 1$, while $\alpha_i \cap \alpha'_j = \emptyset$ if $i \neq j$ (see Figure 4). Similarly, we may assume that β'_i is a small Hamiltonian translate of β_i so that it intersects β_i transversally in exactly two distinct points near the unique intersection point of a'_i and b_i if $i \neq 1$, while $\beta_i \cap \beta'_j = \emptyset$ if $i \neq j$. With the above understood, consider

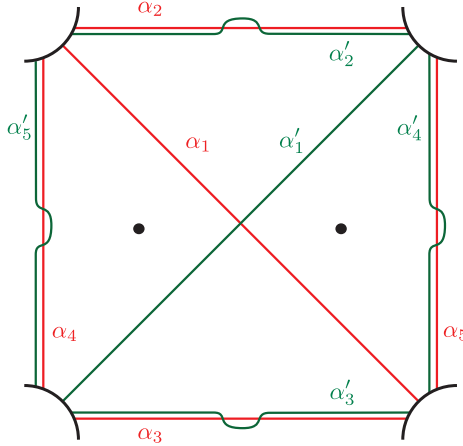


FIGURE 4. The part of the page $S \times \{\frac{1}{2}\}$ depicting the α -curves in red and α' -curves in green. The cases $\alpha_2 = \alpha_3$ and/or $\alpha_4 = \alpha_5$ are allowed.

now the chain map induced by the cobordism described by the multipointed Heegaard triple diagrams $(\Sigma, \beta, \alpha, \alpha', \mathbf{z})$. This cobordism is diffeomorphic to the complement of a tubular neighborhood of a bouquet of N embedded circles in the product cobordism $[0, 1] \times M$. Consequently, it has two incoming boundary components, one of which is diffeomorphic

to $\#_{\mathbb{N}} S^1 \times S^2$, and one outgoing boundary component. We claim that the multipointed Heegaard triple diagram $(\Sigma, \beta, \alpha, \alpha', \mathbf{z})$ is admissible. Recall that a multipointed Heegaard triple diagram $(\Sigma, \beta, \alpha, \alpha', \mathbf{z})$ is admissible if any non-trivial triply periodic domain that can be written as a sum of doubly periodic domains in the multipointed Heegaard diagrams $(\Sigma, \beta, \alpha, \mathbf{z})$, $(\Sigma, \beta, \alpha', \mathbf{z})$, and $(\Sigma, \alpha, \alpha', \mathbf{z})$ has both positive and negative coefficients (see [Lip06, Definition 10.12]). In the present context, note that there exist basepoints on both sides of every curve in the diagram except for α_1 , α_2 , α_3 , β_2 , β_3 , α'_2 , and α'_3 . If a triply periodic domain such as above has α_1 with non-zero multiplicity on its boundary, then it has to have β_1 or α'_1 , or both, with non-zero multiplicity on its boundary as well. In any case, such a triply periodic domain has both negative and positive coefficients. This is because β_1 and α'_1 both admit basepoints on both sides. For example, suppose that both α_1 and α'_1 appear with non-zero multiplicity on the boundary of some triply periodic domain. Then this forces the shaded and dotted regions in Figure 5 to have multiplicities with opposite signs. So suppose now that none of α_1 , β_1 , α'_1 appear with non-zero multiplicity on the boundary of a triply periodic domain. Then we can erase these curves, and apply the same argument to the remaining set of curves noting that each of the remaining curves now has basepoints on both sides. As a result, the multipointed triple Heegaard diagram $(\Sigma, \beta, \alpha, \alpha', \mathbf{z})$ is admissible, and there exists a chain map

$$\hat{f}_{\beta', \alpha, \alpha'; t_{\xi}} : \widehat{CF}(\Sigma, \beta, \alpha, \mathfrak{s}_{\xi}) \otimes_{\mathbb{F}} \widehat{CF}(\Sigma, \alpha, \alpha', \mathfrak{s}_{\circ}) \rightarrow \widehat{CF}(\Sigma, \beta, \alpha', \mathfrak{s}_{\xi}).$$

Let $\vec{\theta}$ denote the top degree generator of $\widehat{CF}(\Sigma, \alpha, \alpha', \mathfrak{s}_{\circ})$ as before. Then it follows similarly to the proof of Proposition 3.2 that the chain map

$$\hat{f}_{\beta, \alpha, \alpha'; t_{\xi}}(\cdot \otimes \vec{\theta}) : \widehat{CF}(\Sigma, \beta, \alpha, \mathfrak{s}_{\xi}) \rightarrow \widehat{CF}(\Sigma, \beta, \alpha', \mathfrak{s}_{\xi}),$$

induces a filtered chain map between the corresponding filtered chain complexes. In this regard, note that the J_+ -filtered differential on $\widehat{CF}(\Sigma, \alpha, \alpha', \mathfrak{s}_{\circ})$ is, once again, identically zero. To be more explicit, $\widehat{CF}(\Sigma, \alpha, \alpha', \mathfrak{s}_{\circ})$ is again isomorphic to $(\mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)})^{\otimes N}$ where $\mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)}$ is a graded module over \mathbb{F} with vanishing differential; however, this time, there are two annular domains with $J_+ = 2$ corresponding to the pseudo-holomorphic curves that contribute to the differential of the generator, $\theta_1 \times [0, 1]$, of $\mathbb{F}_{(1)}$ in a distinguished copy of $\mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)}$. In all other copies of $\mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)}$, the domains corresponding to the pseudo-holomorphic curves contributing to the differential of the generator, $\theta_i \times [0, 1]$ for $i \neq 1$, of $\mathbb{F}_{(1)}$ are bigons as before.

What remains to be shown is that $\hat{f}_{\beta, \alpha, \alpha'; t_{\xi}}(\vec{x} \otimes \vec{\theta}) = \vec{y}$ if we work with a generic split almost complex structure on $\Sigma \times T$. There are two steps to the proof of this claim: the first step shows that there is a single domain that contributes to this chain map, and the second step shows that this domain has a unique pseudo-holomorphic representative.

Step 1. Because of the placement of the basepoints, any holomorphic domain \mathcal{D} representing a class in $\hat{\pi}_2(\vec{x}, \vec{\theta}, \cdot)$ must have multiplicity 1 in the small triangles enclosed by β_i , α_i , and α'_i , ordered in clockwise fashion, for $i > 1$. Next, the fact that \mathcal{D} has corners at x_1 and θ_1 , and that \mathcal{D} has multiplicity zero in the regions with basepoints, implies that \mathcal{D} has multiplicity 1 in the region shaded in Figure 5, and it has multiplicity zero in the dotted region. Adding on regions to form an appropriate holomorphic domain representing

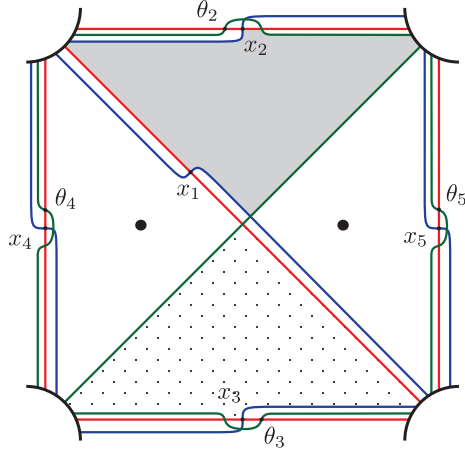


FIGURE 5. The part of the page $S \times \{\frac{1}{2}\}$ depicting the region having one corner at x_1 , and the α -curves in red, β -curves in blue, and α' -curves in green.

a class in $\hat{\pi}_2(\vec{x}, \vec{\theta}, \cdot)$ yields the shaded annular domain depicted in Figure 6. It is easy to verify using Sarkar's Maslov index formula for Whitney n -gons (see [Sar11]) that the Maslov index of this domain is zero.

Step 2. This part establishes existence and uniqueness of a pseudo-holomorphic curve in $\Sigma \times T$ representing the shaded domain in Figure 6. The Riemann Mapping Theorem tells us that a disk in the complex plane with three marked points on its boundary is conformally unique (see [LV73, Chapter 1]). This accounts for the small triangular regions in the shaded domain. As for the shaded annular subdomain, we recall certain facts from complex analysis: the moduli space of annuli with three marked points on each boundary component has dimension six. Such an annulus admits at most one automorphism which exchanges the two boundary components. Topologically, an annulus admits a double branched cover over the disk with two branch points. In particular, an annulus with three marked points on each boundary component admits a branched cover over a disk with three marked points on its boundary if and only if the proportions of the lengths of the three arcs in the complement of the markings on each boundary component agree. Moreover, such a branched cover is unique. Note that the shaded annular subdomain has two obtuse corners depicted on top of Figure 6a. Along the bottom boundary component, which has only acute corners, the lengths of the red and the green arcs are roughly the same, with the red arc slightly shorter, while the length of the blue arc is much shorter. Arrange the curves so that the ratio of the length of the red arc to the length of the blue arc on the top boundary component is bigger than that on the bottom boundary component; in other words, perturb the blue arc on the bottom boundary component until the ratio is a little less than that on the top. This means that in order to get a branched cover, we have to increase the lengths of both the blue and the green arcs on the top boundary component. We can achieve this by cutting a slit along

the blue and the green curves starting from the obtuse corners. Once this is done, we can claim that there is a unique holomorphic curve representing the resulting annular domain.

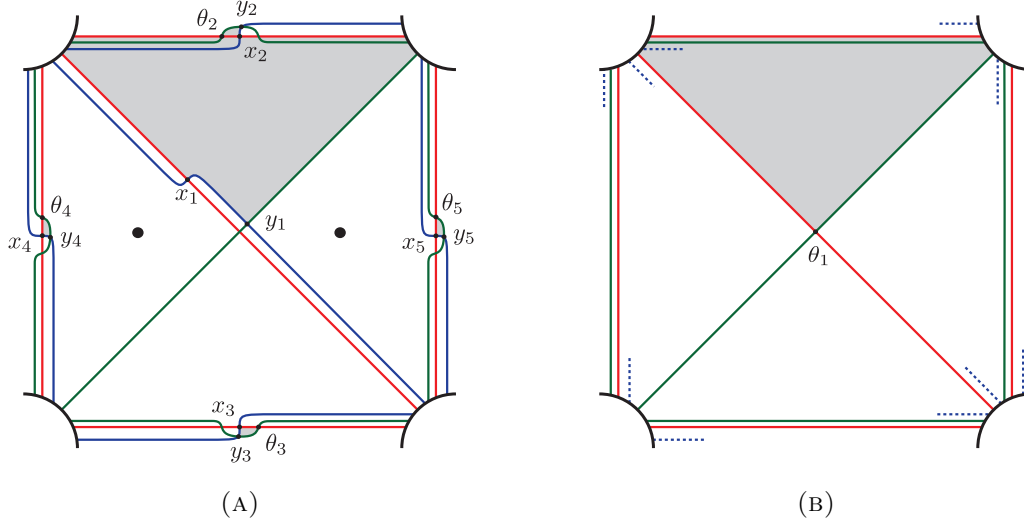


FIGURE 6. The shaded annular domain in the multipointed triple Heegaard diagram $(\Sigma, \beta, \alpha, \alpha', \mathbf{z})$. On the right is the part of the page $S \times \{0\}$ depicting the α -curves in red, β -curves in dotted blue, and α' -curves in green.

Analogous claims for the chain map induced by the cobordism specified by the multipointed Heegaard triple diagram $(\Sigma, \beta', \beta, \alpha', \mathbf{z})$ hold true as well: the multipointed Heegaard triple diagram $(\Sigma, \beta', \beta, \alpha', \mathbf{z})$ is admissible due to the placement of the basepoints, which forces any holomorphic domain representing a class in $\hat{\pi}_2(\vec{\theta}', \vec{\gamma}, \cdot)$ to have multiplicity 1 in the shaded small triangles enclosed by β'_i , β_i , and α'_i ordered in clockwise fashion (see Figure 7), and to have multiplicity zero elsewhere. Then it follows from the Riemann Mapping Theorem that there is a unique pseudo-holomorphic curve representing this domain. As a result, $\hat{f}_{\beta', \beta, \alpha'; t_\xi}(\vec{\theta}' \otimes \vec{\gamma}) = \vec{x}'$. \square

4. PROPERTIES OF \mathbf{o}

This section proves properties of the contact invariant \mathbf{o} summarized in the Introduction. To begin, the claim in the first bullet of Theorem 1.1 follows without much effort thanks to a characterization of overtwisted contact structures via open book decompositions due to Honda, Kazez, and the second author.

Theorem 4.1. *Let ξ_{OT} be an overtwisted contact structure on a closed 3-manifold M . Then $\mathbf{o}(M, \xi_{\text{OT}}) = 0$.*

Proof. Note that an overtwisted contact structure is supported by an open book decomposition (S, ϕ) where the monodromy ϕ is not right-veering [HKM07, Theorem 1.1].

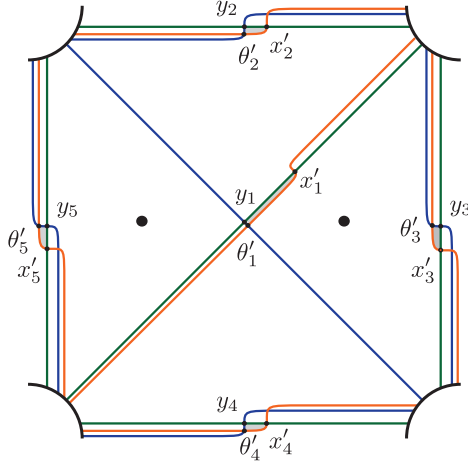


FIGURE 7. The part of the page $S \times \{\frac{1}{2}\}$ depicting the Pachner move with β -curves in blue, α' -curves in green, and β' -curves in orange.

One can find a basis of arcs \mathbf{a} on S so that in the corresponding Heegaard diagram $\widehat{\partial}_{\text{HF}} \vec{\mathbf{y}} = \vec{\mathbf{x}}_{\xi_{\text{OT}}}$ where $\mathbf{y} = \{y_1, x_2, \dots, x_G\}$ and there is exactly one Maslov index-1 holomorphic domain \mathcal{D} , a bigon, that contributes to the differential [HKM09b, Lemma 3.2] as defined by a split complex structure on $\Sigma \times [0, 1] \times \mathbb{R}$. Therefore, $n_{\mathbf{y}}(\mathcal{D}(A)) = \frac{1}{4}$, $n_{\mathbf{x}_{\xi_{\text{OT}}}}(\mathcal{D}) = \frac{1}{4}$, $|\mathbf{y}| = G$, and $|\mathbf{x}_{\xi_{\text{OT}}}| = G$. Applying (2.6), we find $J_+(\mathcal{D}) = 0$. As a result, $\mathbf{o}(M, \xi_{\text{OT}}) = 0$. \square

As for the rest of the claims in the Introduction, first we need to establish a few basic properties of o . To do so, we work in a slightly more general context where we consider arc collections that may not contain a basis. Let (S, ϕ) be an open book decomposition. Given an arc collection \mathbf{a} on S that does not necessarily contain a basis, we can extend it to an arc collection $\tilde{\mathbf{a}}$ that contains a basis. Then we fix a generic almost complex structure J_{HF} for the multi-pointed Heegaard diagram $(\Sigma, \tilde{\beta}, \tilde{\alpha}, \tilde{\mathbf{z}})$ associated to the arc collection $\tilde{\mathbf{a}}$. We may regard $\widehat{CF}(\Sigma, \beta, \alpha)$ as a submodule of $\widehat{CF}(\Sigma, \tilde{\beta}, \tilde{\alpha})$ by identifying the generators of $\widehat{CF}(\Sigma, \beta, \alpha)$ with the generators obtained from these by adding on the distinguished points lying in $S \times \{\frac{1}{2}\}$ for each of the arcs in $\mathbf{a} \setminus \tilde{\mathbf{a}}$. Due to the placement of the basepoints there can be no pseudo-holomorphic curves with negative punctures at the chords resulting from these points. Therefore, the differential on $\widehat{CF}(\Sigma, \beta, \alpha)$ and on the submodule of $\widehat{CF}(\Sigma, \tilde{\beta}, \tilde{\alpha})$ that it is identified with coincide. As a result, we may consider $\widehat{CF}(\Sigma, \beta, \alpha)$ as a subcomplex of $\widehat{CF}(\Sigma, \tilde{\beta}, \tilde{\alpha})$. With the preceding understood, the first basic property of o is that it is non-increasing under enlargement of arc collections.

Lemma 4.2. *Suppose that $\mathbf{a}_1 \subset \mathbf{a}_2$ are two collections of pairwise disjoint properly embedded arcs on S . Then there exists a generic almost complex structure J_{HF} on $\Sigma \times [0, 1] \times \mathbb{R}$, and an inclusion of chain complexes*

$$I : \widehat{CF}(\Sigma, \beta_1, \alpha_1) \rightarrow \widehat{CF}(\Sigma, \beta_2, \alpha_2),$$

$$\mathcal{I} : \widehat{\mathcal{CF}}(S, \phi, \mathbf{a}_1) \rightarrow \widehat{\mathcal{CF}}(S, \phi, \mathbf{a}_2),$$

such that the contact generator is mapped to the contact generator by the first inclusion, while the latter inclusion induces a morphism of spectral sequences from $E^*(S, \phi, \mathbf{a}_1; J_{HF})$ to $E^*(S, \phi, \mathbf{a}_2; J_{HF})$; hence, $o(S, \phi, \mathbf{a}_1; J_{HF}) \geq o(S, \phi, \mathbf{a}_2; J_{HF})$.

Proof. It suffices to find a generic almost complex structure J_{HF} on $\Sigma \times [0, 1] \times \mathbb{R}$ so that moduli spaces of J_{HF} -holomorphic curves associated to the Heegaard diagram $(\Sigma, \beta_2, \alpha_2)$ are cut out transversally, because this immediately implies transversality of moduli spaces of J_{HF} -holomorphic curves associated to the Heegaard diagram $(\Sigma, \beta_1, \alpha_1)$. Having fixed such a generic almost complex structure, the inclusion map I is defined on the set of generators of $\widehat{\mathcal{CF}}(\Sigma, \beta_1, \alpha_1)$ by

$$I(\vec{y}) = \vec{y}'$$

where $\mathbf{y}' = \mathbf{y} \cup \{x_a\}_{a \in \mathbf{a}_2 \setminus \mathbf{a}_1}$ and x_a is the unique intersection point of a and b for an arc $a \in \mathbf{a}_2 \setminus \mathbf{a}_1$. It follows that $I(\vec{x}_\xi^1) = \vec{x}_\xi^2$. Meanwhile, the J_{HF} -holomorphic curves that define the differential acting on elements of the subgroup $I(\widehat{\mathcal{CF}}(\Sigma, \beta_1, \alpha_1))$ are the same as the J_{HF} -holomorphic curves that define the differential on $\widehat{\mathcal{CF}}(\Sigma, \beta_1, \alpha_1)$. Therefore, I is a chain map, and the induced inclusion map \mathcal{I} is a filtered chain map. The latter induces a morphism of spectral sequences from $E^*(S, \phi, \mathbf{a}_1; J_{HF})$ to $E^*(S, \phi, \mathbf{a}_2; J_{HF})$; hence, $o(S, \phi, \mathbf{a}_1; J_{HF}) \geq o(S, \phi, \mathbf{a}_2; J_{HF})$. \square

The next lemma claims that o remains the same under suitable enlargement of the pages of an open book decomposition while keeping the arc collection untouched.

Lemma 4.3. *Let \mathbf{a} be a collection of pairwise disjoint properly embedded arcs on S , and S' be a compact oriented surface with boundary obtained from S by attaching 1-handles away from a neighborhood of $\partial \mathbf{a}$. Let $\phi' : S' \rightarrow S'$ be an orientation-preserving diffeomorphism whose restriction to \mathbf{a} agrees with ϕ . Then there are generic almost complex structures J_{HF} and J'_{HF} to define the differentials on $\widehat{\mathcal{CF}}(\Sigma, \beta, \alpha)$ and $\widehat{\mathcal{CF}}(\Sigma', \beta, \alpha)$, respectively, such that $(\widehat{\mathcal{CF}}(S, \phi, \mathbf{a}), \widehat{\partial})$ and $(\widehat{\mathcal{CF}}(S', \phi', \mathbf{a}), \widehat{\partial}')$ are isomorphic as filtered chain complexes. As a result, $o(S, \phi, \mathbf{a}; J_{HF}) = o(S', \phi', \mathbf{a}; J'_{HF})$.*

Proof. It follows from the description of the surface S' that \mathbf{a} can also be seen as a pairwise disjoint collection of properly embedded arcs on S' . Moreover, there is a canonical 1–1 correspondence between unordered tuples of intersection points in the Heegaard diagrams (Σ, β, α) and (Σ', β, α) . Also note that Σ' is obtained from Σ by connected summing with tori along regions in the Heegaard diagram (Σ, β, α) with basepoints. Therefore, having fixed a generic almost complex structure J_{HF} on $\Sigma \times [0, 1] \times \mathbb{R}$, we can “extend” it to a generic almost complex structure J'_{HF} on $\Sigma' \times [0, 1] \times \mathbb{R}$ so that the holomorphic domains in the pointed Heegaard diagrams $(\Sigma, \beta, \alpha, \mathbf{z})$ and $(\Sigma', \beta, \alpha, \mathbf{z})$ agree, and the claim follows. \square

A key application of these lemmas is to the behavior of \circ (see Definition 2.3) under positive stabilization. Recall that a positive stabilization of an open book decomposition (S, ϕ) is an open book decomposition (S', ϕ') where S' is obtained from S by attaching a 1-handle H , and ϕ' differs from ϕ by a right-handed Dehn twist around a simple closed curve $c \subset S'$ that intersects the cocore of H in exactly one point; in other words, $\phi' = \phi \circ \tau_c$.

Corollary 4.4. *Let $\mathcal{B} := (S, \phi)$ be an open book decomposition, and $\mathcal{B}' := (S', \phi')$ be a positive stabilization of \mathcal{B} . Then $\mathfrak{o}(\mathcal{B}') \leq \mathfrak{o}(\mathcal{B})$.*

Proof. This follows from Lemmas 4.2 and 4.3. To be more explicit, configure \mathcal{B}' as $S' = S \cup H$ and $\phi' = \phi \circ \tau_c$ where c is a curve in S' that intersects the co-core of the 1-handle H in exactly one point. Now fix a complete set of arcs \mathbf{a} on S disjoint from c , and complete it to a complete set of arcs \mathbf{a}' on S' . Then by Lemma 4.2

$$\mathfrak{o}(\mathcal{B}') = o(S', \phi', \mathbf{a}') \leq o(S', \phi', \mathbf{a}),$$

while

$$o(S', \phi', \mathbf{a}) = o(S, \phi, \mathbf{a}) = \mathfrak{o}(\mathcal{B}),$$

by Lemma 4.3. This finishes the proof. \square

Having established some basic properties of o , we are now ready to prove our claim that $\mathfrak{o}^c = \mathfrak{o}$.

Proposition 4.5. *Let (M, ξ) be a closed contact 3-manifold. Then $\mathfrak{o}(M, \xi) = \mathfrak{o}^c(M, \xi)$.*

Proof. It immediately follows from the definitions of \mathfrak{o} and \mathfrak{o}^c that

$$\mathfrak{o}(M, \xi) \leq \mathfrak{o}^c(M, \xi).$$

To prove the reverse inequality, namely, $\mathfrak{o}(M, \xi) \geq \mathfrak{o}^c(M, \xi)$, fix an open book decomposition $\mathcal{B} = (S, \phi)$ and a pairwise disjoint properly embedded collection of arcs \mathbf{a} that contains a basis such that $\mathfrak{o}(M, \xi) = o(S, \phi, \mathbf{a})$. If the arc collection \mathbf{a} can be enlarged to a complete set of arcs \mathbf{a}' , the claim follows since by Lemma 4.2

$$o(S, \phi, \mathbf{a}) \geq o(S, \phi, \mathbf{a}') = \mathfrak{o}(\mathcal{B}) \geq \mathfrak{o}^c(M, \xi).$$

Otherwise, we can positively stabilize the open book decomposition \mathcal{B} sufficiently many times to obtain an open book decomposition $\mathcal{B}' = (S', \phi')$ of M supporting ξ so that \mathbf{a} can be enlarged to a complete set of arcs \mathbf{a}' on S' . In fact, this can be done in such a way that the effect of the monodromy ϕ' on arcs from \mathbf{a} agrees with the effect of the monodromy ϕ . To achieve this, all we have to do is to separate isotopic pairs of arcs a_i and a_j in \mathbf{a} by positively stabilizing \mathcal{B} as in Figure 8. Since \mathbf{a} contains a basis of arcs, the complement of \mathbf{a} in S is a disjoint union of disks, and each time we positively stabilize \mathcal{B} and add the co-core a' of the attached 1-handle H , we maintain this property. Once we reach a collection of pairwise non-isotopic arcs which cuts the surface S' into a disjoint union of disks, we can enlarge it to a complete set of arcs \mathbf{a}' on S' . Then, by Lemmas 4.2 and 4.3, we have

$$o(S, \phi, \mathbf{a}) = o(S', \phi', \mathbf{a}) \geq o(S', \phi', \mathbf{a}') = \mathfrak{o}(\mathcal{B}') \geq \mathfrak{o}^c(M, \xi),$$

which completes the proof. \square

We move on to analyze the behavior of \mathfrak{o} under Legendrian surgery.

Proposition 4.6. *Let (S, ϕ) be an open book decomposition and \mathbf{a} be any collection of pairwise disjoint properly embedded arcs on S that contains a basis. Suppose c is a non-separating simple closed curve on S which meets any curve in $\phi(\mathbf{a})$ at most once. Then $o(S, \tau_c \circ \phi, \mathbf{a}) \geq o(S, \phi, \mathbf{a})$.*

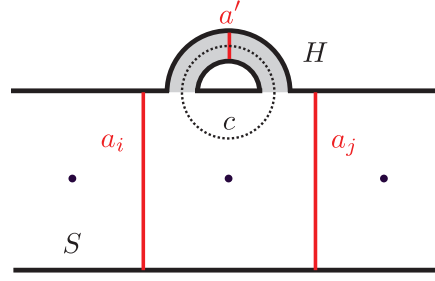


FIGURE 8. Stabilizing the open book decomposition (S, ϕ) to get a new open book decomposition $(S' = S \cup H, \phi' = \phi \circ \tau_c)$.

Proof. To start with, use (S, ϕ, \mathbf{a}) and the curve c to form a multipointed triple Heegaard diagram $(\Sigma, \beta, \gamma, \alpha, \mathbf{z})$ where $\gamma = \{\gamma_1, \dots, \gamma_k\}$ with $\gamma_i = b'_i \times \{\frac{1}{2}\} \cup \tau_c \circ \phi(b'_i) \times \{0\}$ such that b'_i is obtained from b_i by slightly pushing along ∂S in the direction of the boundary orientation as in Figure 9.

Notice that $(\Sigma, \beta, \alpha, \mathbf{z})$ is the multipointed Heegaard diagram associated to (S, ϕ, \mathbf{a}) and $(\Sigma, \gamma, \alpha, \mathbf{z})$ is the multipointed Heegaard diagram associated to $(S, \tau_c \circ \phi, \mathbf{a})$. Meanwhile, the multipointed Heegaard diagram (Σ, β, γ) describes the manifold $\#_{k-1} S^1 \times S^2$. Note also that by the open book decomposition (S, τ_c) together with the collection of arcs $\{b_1, \dots, b_k\}$ specifies the Heegaard diagram (Σ, γ, β) as in [HKM09b]. The chain complex $\widehat{CF}(\Sigma, \beta, \gamma)$ has trivial differential and the generator $\tilde{\theta}$ indicated in Figure 9 is the topmost generator.

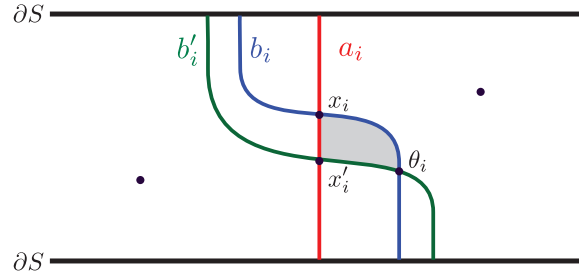


FIGURE 9. Part of the restriction of the multipointed Heegaard triple diagram $(\Sigma, \beta, \gamma, \alpha, \mathbf{z})$ to $S \times \{\frac{1}{2}\} \subset \Sigma$.

The placement of the basepoints guarantees, once again, that the multipointed triple Heegaard diagram $(\Sigma, \beta, \gamma, \alpha, \mathbf{z})$ is admissible. Now consider the chain map

$$\hat{f}_{\beta, \gamma, \alpha} : \widehat{CF}(\Sigma, \beta, \gamma) \otimes_{\mathbb{F}} \widehat{CF}(\Sigma, \gamma, \alpha) \rightarrow \widehat{CF}(\Sigma, \beta, \alpha), \quad (4.1)$$

induced by the cobordism described by the triple Heegaard diagram $(\Sigma, \beta, \gamma, \alpha)$. This chain map is non-trivial only in the canonical Spin^c structure \mathfrak{t}_ω corresponding to the Stein structure ω on the Legendrian surgery cobordism. This Spin^c structure restricts to the canonical Spin^c structures \mathfrak{s}_ξ and $\mathfrak{s}_{\xi'}$ before and after surgery, respectively. Moreover, as is depicted in Figure 10, the differential on $\mathbb{F} \cdot \tilde{\theta} \otimes_{\mathbb{F}} \widehat{CF}(\Sigma, \gamma, \alpha, \mathfrak{s}_{\xi'})$ is identically zero; hence,

it is a subcomplex of $\widehat{CF}(\Sigma, \beta, \gamma) \otimes_{\mathbb{F}} \widehat{CF}(\Sigma, \gamma, \alpha)$. Restricting (4.1) to this subcomplex, we obtain a chain map

$$\hat{f}_{\beta, \gamma, \alpha; t_\omega}(\vec{\theta} \otimes \cdot) : \widehat{CF}(\Sigma, \gamma, \alpha, s_{\xi'}) \rightarrow \widehat{CF}(\Sigma, \beta, \alpha, s_{\xi}).$$

In fact, the J_+ -filtered differential on $\mathbb{F} \cdot \vec{\theta} \otimes_{\mathbb{F}} \widehat{CF}(\Sigma, \gamma, \alpha, s_{\xi'})$ is identically zero since all homology classes in $\hat{\pi}_2(\vec{\theta}, \cdot)$ have the same J_+ value (see Figure 10). Therefore, having decomposed the above chain map as

$$\hat{f}_{\beta, \gamma, \alpha; t_\omega}(\vec{\theta} \otimes \cdot) = f^0 + f^1 + \cdots + f^\ell + \cdots,$$

where f^ℓ counts embedded Fredholm index-0 pseudo-holomorphic curves with $J_+ = 2\ell$, we have

$$\sum_{i+j=\ell} (f^i \circ \partial'_j - \partial_i \circ f^j) = 0. \quad (4.2)$$

just as in Section 3. The identity (4.2) implies that there is a filtered chain map from $(\widehat{CF}(S, \tau_c \circ \phi, \mathbf{a}), \hat{\partial}')$ to $(\widehat{CF}(S, \phi, \mathbf{a}), \hat{\partial})$ and hence a morphism of spectral sequences from $E^*(S, \tau_c \circ \phi, \mathbf{a}; J'_{HF})$ to $E^*(S, \phi, \mathbf{a}; J_{HF})$. In addition, $\hat{f}_{\beta, \gamma, \alpha; t_\omega}(\vec{\theta} \otimes \vec{x}'_\xi) = \vec{x}_\xi$ since the shaded triangle in Figure 9 is the only holomorphic domain that contributes to this chain map due to the placement of the basepoints, and it is represented by a unique pseudo-holomorphic curve by the Riemann Mapping Theorem. Hence, $o(S, \tau_c \circ \phi, \mathbf{a}; J'_{HF}) \geq o(S, \phi, \mathbf{a}; J_{HF})$ as desired. \square

Corollary 4.7. *Let $\mathcal{B} := (S, \phi)$ be an open book decomposition and suppose $\mathcal{B}' := (S, \phi')$ is obtained from \mathcal{B} by Legendrian surgery, i.e. $\phi' = \tau_{c_n} \circ \cdots \circ \tau_{c_1} \circ \phi$. Then*

$$\mathfrak{o}(\mathcal{B}) \leq \mathfrak{o}(\mathcal{B}'). \quad (4.3)$$

As a consequence, if $\mathcal{B} := (S, \phi)$ is an open book decomposition where ϕ can be written as a product of positive Dehn twists, then $\mathfrak{o}(\mathcal{B}) = \infty$.

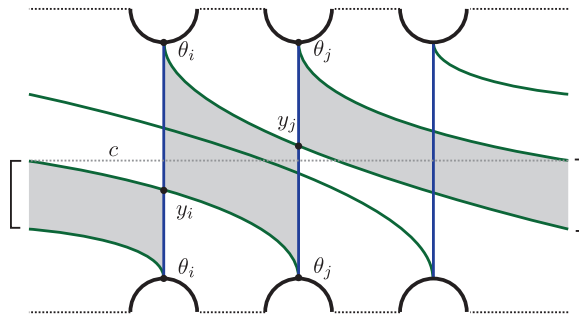


FIGURE 10. A local picture of $-S \times \{0\} \subset \Sigma$ part of the Heegaard diagram (Σ, β, γ) near the surgery curve and all intersecting arcs. The shaded domains representing pseudo-holomorphic curves with negative punctures at $\vec{\theta}$ have the same J_+ value. The brackets indicate that the ends of the shaded region connect to one another.

Proof. To prove the inequality, apply Proposition 4.6 one Dehn twist at a time, noting that for each Dehn twist curve c_i we can find a complete set of arcs \mathbf{a} on S so that c_i intersects each arc in $\phi(\mathbf{a})$ at most once. In between successive Legendrian surgeries, apply Proposition 3.3 to switch to a suitable complete set of arcs. The last claim of the corollary follows immediately from (4.3) once we note that $\mathfrak{o}(S, id_S) = \infty$. The latter is because the J_+ -filtered differential in the corresponding Heegaard Floer chain complex is zero. \square

Now we can prove Theorem 1.2, that spectral order is non-decreasing under Legendrian surgery:

Proof of Theorem 1.2. Let (M', ξ') be obtained from (M, ξ) by Legendrian surgery. In light of Proposition 4.5, we shall prove that $\mathfrak{o}^c(M, \xi) \leq \mathfrak{o}^c(M', \xi')$. It suffices to prove this for Legendrian surgery on a single curve. Suppose without loss of generality that the Legendrian c lies on a page of an open book decomposition \mathcal{B} of M supporting ξ . Positively stabilize \mathcal{B} to get to an open book decomposition \mathcal{B}_1 which realizes $\mathfrak{o}^c(M, \xi)$; namely, $\mathfrak{o}(\mathcal{B}_1) = \mathfrak{o}^c(M, \xi)$. Now consider the open book decomposition \mathcal{B}_2 of M' supporting ξ' obtained by Legendrian surgery on c . Positively stabilize \mathcal{B}_2 to get an open book decomposition \mathcal{B}' with $\mathfrak{o}(\mathcal{B}') = \mathfrak{o}^c(M', \xi')$. Now, mirroring these stabilizations on \mathcal{B}_1 , we obtain an open book decomposition \mathcal{B}'_1 of M supporting ξ which, after Legendrian surgery, gives \mathcal{B}' . By Corollary 4.4 we have $\mathfrak{o}(\mathcal{B}'_1) \leq \mathfrak{o}(\mathcal{B}_1) = \mathfrak{o}^c(M, \xi)$, which implies that $\mathfrak{o}(\mathcal{B}'_1) = \mathfrak{o}^c(M, \xi)$; and by Corollary 4.7 we have $\mathfrak{o}(\mathcal{B}'_1) \leq \mathfrak{o}(\mathcal{B}') = \mathfrak{o}^c(M', \xi')$. Hence

$$\mathfrak{o}^c(M, \xi) = \mathfrak{o}(\mathcal{B}'_1) \leq \mathfrak{o}(\mathcal{B}') = \mathfrak{o}^c(M', \xi').$$

\square

With all the results needed in place, we are ready to prove the claim in the second bullet of Theorem 1.1.

Corollary 4.8. *Let (M, ξ) be Stein fillable. Then $\mathfrak{o}(M, \xi) = \infty$.*

Proof. A Stein fillable contact 3-manifold admits a supporting open book decomposition (S, ϕ) where ϕ is a product of positive Dehn twists. To be more explicit, a Stein fillable contact 3-manifold can be obtained via Legendrian surgery on some connected sum $\#_N S^1 \times S^2$ equipped with its standard contact structure ξ_{std} (see [Gom98]). Therefore, by Theorem 1.2 and Proposition 4.5, it suffices to prove that $\mathfrak{o}^c(\#_N S^1 \times S^2, \xi_{std}) = \infty$. To see this, let \mathcal{B} be an open book decomposition of $\#_N S^1 \times S^2$ supporting ξ_{std} which realizes $\mathfrak{o}^c(\#_N S^1 \times S^2, \xi_{std})$; in other words, $\mathfrak{o}(\mathcal{B}) = \mathfrak{o}^c(\#_N S^1 \times S^2, \xi_{std})$. As $(\#_N S^1 \times S^2, \xi_{std})$ is supported by an open book with trivial monodromy, a common stabilization, \mathcal{B}' , of that and \mathcal{B} will have a monodromy which can be written as a product of positive Dehn twists and will also realize the minimal \mathfrak{o} . To see this, note that by the second claim in Corollary 4.7, we have $\mathfrak{o}(\mathcal{B}') = \infty$. By Corollary 4.4, we also have $\mathfrak{o}(\mathcal{B}') \leq \mathfrak{o}(\mathcal{B}) = \mathfrak{o}^c(\#_N S^1 \times S^2, \xi_{std})$. Therefore, $\mathfrak{o}^c(\#_N S^1 \times S^2, \xi_{std}) = \infty$. \square

Finally, we prove the claim in the third bullet of Theorem 1.1. Loosely speaking, the spectral order can be computed from any open book decomposition of M supporting ξ ,

and any complete set of arcs, by adding one additional arc per boundary component of S , subject to certain constraints.

Theorem 4.9. *Let (S, ϕ) be an open book decomposition of M supporting ξ , \mathbf{a} be a complete set of arcs on S , and $\tilde{\mathbf{a}}$ be a collection of \mathbb{B} arcs from \mathbf{a} , possibly with repetition, such that there is a bijective map from the set of connected components of ∂S to $\tilde{\mathbf{a}}$ which sends a component of ∂S to an arc in $\tilde{\mathbf{a}}$ that it intersects. Let $\tilde{\mathbf{a}}$ be the set of arcs obtained from \mathbf{a} by adding a parallel copy of each arc from \mathbf{a} . Then*

$$o(S, \phi, \tilde{\mathbf{a}}) = \mathbf{o}(M, \xi).$$

Proof. Let $\mathcal{B} = (S, \phi)$ be an open book decomposition of M supporting ξ . We positively stabilize it to get to an open book decomposition $\mathcal{B}' = (S', \phi')$ with S' built from S by adding 1-handles, and $\phi' = \tau_{c_n} \circ \dots \circ \tau_{c_1} \circ \phi$, such that \mathcal{B}' realizes $\mathbf{o}^c(M, \xi)$; that is, $\mathbf{o}(\mathcal{B}') = \mathbf{o}^c(M, \xi)$. Let a_1, \dots, a_m be a minimal collection of pairwise disjoint properly embedded arcs in S' that separate S' into two surfaces one of which contains all the 1-handles and the other, denoted S° , is homeomorphic to S . Extending ϕ to S' as the identity on all the 1-handles, we form the open book decomposition $\tilde{\mathcal{B}} = (S', \phi)$. Since ϕ' is obtained from ϕ by adding positive Dehn twists, $\mathbf{o}(\tilde{\mathcal{B}}) \leq \mathbf{o}(\mathcal{B}')$ by Corollary 4.7.

Now, fix a collection of arcs \mathbf{a} as in the hypothesis of the theorem and a complete set of arcs \mathbf{a} on S contained in S° , and let $\tilde{\mathbf{a}}$ be the set of arcs obtained from \mathbf{a} by adding a parallel copy of each arc from \mathbf{a} . Extend $\tilde{\mathbf{a}}$ to a complete set of arcs \mathbf{a}' on S' which includes the arcs a_1, \dots, a_m . Then we can isotope \mathbf{a}' so that $\mathbf{a}' \cap S = \tilde{\mathbf{a}}$ and note that ϕ acts trivially on all arcs in $\mathbf{a}' \setminus \tilde{\mathbf{a}}$. Looking at the Heegaard diagram resulting from (S', ϕ, \mathbf{a}') , the α and β curves corresponding to arcs in $S' \setminus S$ intersect each other exactly twice, forming two canceling bigons and thus contribute zero to $\hat{\partial}_{HF}$. Furthermore, α_i and β_i intersect no other α -curves or β -curves. Thus

$$\widehat{\mathcal{CF}}(S', \phi, \mathbf{a}') \equiv \widehat{\mathcal{CF}}(S', \phi, \tilde{\mathbf{a}}) \otimes_{\mathbb{F}} (\mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)})^{\otimes n},$$

where $\mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)}$ is a graded module over \mathbb{F} with vanishing differential and n is the number of arcs in \mathbf{a}' contained in $S' \setminus S$. In particular,

$$o(S', \phi, \mathbf{a}') = o(S', \phi, \tilde{\mathbf{a}})$$

By Lemma 4.3, we have $o(S, \phi, \tilde{\mathbf{a}}) = o(S', \phi, \tilde{\mathbf{a}})$. Consequently,

$$o(S, \phi, \tilde{\mathbf{a}}) = o(S', \phi, \tilde{\mathbf{a}}) = o(S', \phi, \mathbf{a}') \leq o(S', \phi', \mathbf{a}') = \mathbf{o}^c(M, \xi) = \mathbf{o}(M, \xi).$$

On the other hand, the reverse inequality follows from Proposition 4.5:

$$o(S, \phi, \tilde{\mathbf{a}}) \geq \mathbf{o}(M, \xi).$$

This completes the proof. □

Another application of the Legendrian surgery statement in Theorem 1.2 is our Theorem 1.3, that the spectral order of a contact connected sum is the minimum of the orders of the summands:

Proof of Theorem 1.3. Let $\mathcal{B}_1 = (S_1, \phi_1)$ and $\mathcal{B}_2 = (S_2, \phi_2)$ be open book decompositions which realize $\mathbf{o}^c(M_1, \xi_1)$ and $\mathbf{o}^c(M_2, \xi_2)$, respectively. Fix complete sets of arcs \mathbf{a}_1 and \mathbf{a}_2 for \mathcal{B}_1 and \mathcal{B}_2 , respectively. Then both $\widehat{\mathcal{CF}}(S_1, \phi_1, \mathbf{a}_1)$ and $\widehat{\mathcal{CF}}(S_2, \phi_2, \mathbf{a}_2)$ can be seen as filtered subcomplexes of $\widehat{\mathcal{CF}}(S_\#, \phi_\#, \mathbf{a}_\#)$ where $\mathcal{B}_1 \# \mathcal{B}_2 = (S_\#, \phi_\#)$ is the boundary connected sum open book decomposition with $\phi_\# = \phi_2 \circ \phi_1$, where we extend each by the identity across the complementary subsurface, and $\mathbf{a}_\#$ is a complete set of arcs on $S_\#$ extending $\mathbf{a}_1 \sqcup \mathbf{a}_2$. Hence,

$$\mathbf{o}^c(M_1 \# M_2, \xi_1 \# \xi_2) \leq \mathfrak{o}(\mathcal{B}_1 \# \mathcal{B}_2) \leq \mathfrak{o}(\mathcal{B}_i) = \mathbf{o}^c(M_i, \xi_i),$$

for both $i = 1$ and $i = 2$.

For the reverse inequality, suppose that $\tilde{\mathcal{B}}$ is an open book decomposition of $M_1 \# M_2$ realizing $\mathbf{o}^c(M_1 \# M_2, \xi_1 \# \xi_2)$, and let $\mathcal{B} = (S, \phi)$ be a common stabilization of both $\tilde{\mathcal{B}}$ and $\mathcal{B}_1 \# \mathcal{B}_2$. Ignore the extra positive Dehn twists on \mathcal{B} which arise from its description as a positive stabilization of $\mathcal{B}_1 \# \mathcal{B}_2$. The resulting open book decomposition $\mathcal{B}' = (S, \phi')$ describes the 3-manifold $M_1 \# M_2 \#_k S^1 \times S^2$, the page S contains $S_1 \# S_2$ as a subsurface due to \mathcal{B} being a positive stabilization of $\mathcal{B}_1 \# \mathcal{B}_2$, and the monodromy ϕ' extends $\phi_\#$ as the identity to the rest of S' . In particular, \mathcal{B} is obtained from \mathcal{B}' by Legendrian surgery along curves contained in a page of \mathcal{B} ; hence,

$$\mathbf{o}^c(M_1 \# M_2, \xi_1 \# \xi_2) = \mathfrak{o}(\mathcal{B}) \geq \mathfrak{o}(\mathcal{B}'),$$

by Theorem 1.2.

Looking at \mathcal{B}' , we can choose a complete set of arcs \mathbf{a}' on \mathcal{B}' which includes the separating arc in the boundary connected sum. Then

$$\widehat{\mathcal{CF}}(S, \phi', \mathbf{a}') \cong \widehat{\mathcal{CF}}(S_1, \phi_1, \mathbf{a}_1) \otimes_{\mathbb{F}} \widehat{\mathcal{CF}}(S_2, \phi_2, \mathbf{a}_2) \otimes_{\mathbb{F}} (\mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)})^{\otimes k},$$

as filtered complexes where $\mathbb{F}_{(0)} \oplus \mathbb{F}_{(1)}$ is a graded module over \mathbb{F} with vanishing differential. As a result, $\min\{o(S_1, \phi_1, \mathbf{a}_1), o(S_2, \phi_2, \mathbf{a}_2)\} = \mathfrak{o}(\mathcal{B}') \leq \mathbf{o}^c(M_1 \# M_2, \xi_1 \# \xi_2)$. On the other hand, since $\mathfrak{o}(\mathcal{B}_1)$ and $\mathfrak{o}(\mathcal{B}_2)$ are minimal, $o(S_1, \phi_1, \mathbf{a}_1) = \mathfrak{o}(\mathcal{B}_1) = \mathbf{o}^c(M_1, \xi_1)$ and $o(S_2, \phi_2, \mathbf{a}_2) = \mathfrak{o}(\mathcal{B}_2) = \mathbf{o}^c(M_2, \xi_2)$, hence

$$\min\{\mathbf{o}^c(M_1, \xi_1), \mathbf{o}^c(M_2, \xi_2)\} \leq \mathbf{o}^c(M_1 \# M_2, \xi_1 \# \xi_2).$$

Applying Proposition 4.5 completes the proof. \square

Corollary 4.10. *For any surface S with boundary, the set of monodromies yielding open book decompositions supporting contact 3-manifolds (M, ξ) with $\mathbf{o}(M, \xi) \geq k$ forms a monoid in the mapping class group $\text{Mod}(S, \partial S)$.*

We use $\mathbf{o}^k(S)$ to denote this monoid.

Proof. By [BEVHM12], for any two mapping classes ϕ_1 and ϕ_2 , there is a Stein cobordism starting at the disconnected contact manifold $(M_{\phi_1}, \xi_{\phi_1}) \sqcup (M_{\phi_2}, \xi_{\phi_2})$ and ending at $(M_{\phi_2 \circ \phi_1}, \xi_{\phi_2 \circ \phi_1})$. By Theorems 1.2 and 1.3, this implies that $\mathbf{o}(M_{\phi_2 \circ \phi_1}, \xi_{\phi_2 \circ \phi_1}) \geq \mathbf{o}((M_{\phi_1}, \xi_{\phi_1}) \sqcup (M_{\phi_2}, \xi_{\phi_2})) = \min\{\mathbf{o}(M_{\phi_1}, \xi_{\phi_1}), \mathbf{o}(M_{\phi_2}, \xi_{\phi_2})\}$. \square

5. DETECTING NON-ZERO \mathbf{o}

In this section, we relate spectral order to the framework of [Wan15a]. We use this relation to develop a criterion for non-vanishing of \mathbf{o} which can be read from an arbitrary open book decomposition. To start, we recall some of the framework of [Wan15a], adapted slightly to the context of this article.

Throughout this section, (S, ϕ) is an open book decomposition and \mathbf{a} is a pairwise disjoint properly embedded collection of arcs on S . We will assume that ϕ is such that it maps each arc to the right at each endpoint, and otherwise \mathbf{a} and $\phi(\mathbf{a})$ are minimally intersecting; equivalently, any bigon region in $S \setminus (\mathbf{a} \cup \phi(\mathbf{a}))$ has a corner on ∂S . We will also denote by R_z the union of all components of $S \setminus (\mathbf{a} \cup \phi(\mathbf{a}))$ which have edges along ∂S . Note that, for the case that \mathbf{a} is a basis, R_z is exactly the region with basepoint in the corresponding pointed Heegaard diagram.

Definition 5.1. Let (S, ϕ) be an open book decomposition and \mathbf{a} be a pairwise disjoint properly embedded collection of arcs in S that contains a basis. A *boundary-based region* in (S, ϕ, \mathbf{a}) is the image of an immersion $f : D \looparrowright S$, where D is a $2n$ -gon, whose edges we label $e_0, e_1, \dots, e_{2n-1}$ in counter-clockwise order, and such that:

- (1) For j even, $f|_{e_j}$ is an embedding into some $a \in \mathbf{a}$, while for j odd, $f|_{e_j}$ is an embedding into some $\phi(a)$, for $a \in \mathbf{a}$,
- (2) each corner of $f(D)$ is acute,
- (3) for each even j , $f(e_j \cap e_{j+1}) \subset \partial S$ (taking indices mod $2n$), and
- (4) $f(D) \cup R_z$ does not contain any connected component of $S \setminus \mathbf{a}$ or $S \setminus \phi(\mathbf{a})$.

We denote the set of all such regions by $\mathcal{R}(S, \phi, \mathbf{a})$. Some examples are indicated in Figure 11.

Let (S, ϕ, \mathbf{a}) be as above, R a boundary-based region, and $a_1, a_2 \in \mathbf{a}$ parallel arcs. Notice then that one may ‘shift’ any edge of R along a_1 to one along a_2 as follows: letting Δ denote the disc component of $S \setminus \{a_1, a_2\}$ (Figure 12a), modify R by adding/removing a 4-gon of Δ , and removing/adding a 4-gon of $\phi(\Delta)$ (Figure 12b). We say the resulting boundary-based region $R' \in \mathcal{R}(S, \phi, \mathbf{a} \setminus \{a_1\})$ is obtained from R by an *elementary collapse*.

Definition 5.2. Let (S, ϕ, \mathbf{a}) be as above. A *level 0, positive region* in (S, ϕ, \mathbf{a}) is an element $R \in \mathcal{R}(S, \phi, \mathbf{a})$ which is obtained by elementary collapses of a boundary-based region in $\mathcal{R}(S, \phi, \mathbf{a}')$, all of whose corners are isolated, where \mathbf{a}' is obtained from \mathbf{a} by adding parallel arcs.

We denote the set of all such regions by $\mathcal{R}_0^+(S, \phi, \mathbf{a})$. Figure 11b illustrates an element of $\mathcal{R}_0^+(S, \phi, \mathbf{a})$ in the example of Figure 2.

As for the relevance to this article, we have:

Lemma 5.3. *Let \mathcal{D} be a domain in the multipointed Heegaard diagram corresponding to (S, ϕ, \mathbf{a}) representing a class $A \in \hat{\pi}_2(\vec{\mathbf{y}}, \vec{\mathbf{x}}_\xi)$ with $\mu(A) = 1$ and $J_+(A) = 0$. Then $\mathcal{D} \in \mathcal{R}_0^+(S, \phi, \mathbf{a})$.*

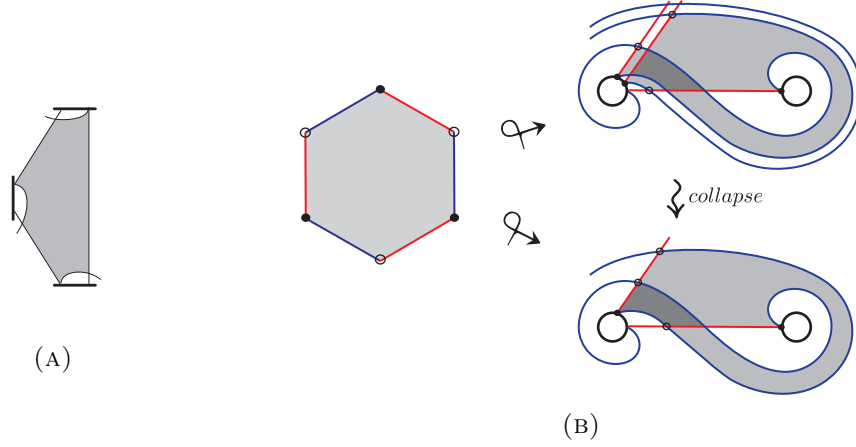


FIGURE 11. To the left, the shaded region is not a boundary-based region, as it does not satisfy condition (4). To the right, two level-0, positive regions related by an elementary collapse. This figure introduces the convention, to be followed throughout the remainder of the paper, that elements of a given arc collection \mathbf{a} are drawn as straight lines, their images under a diffeomorphism curved, while in any figure with multiple line weights, the thickest is reserved for the boundary of a surface.

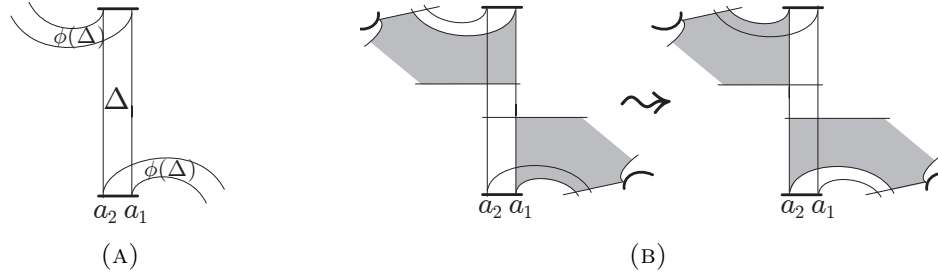


FIGURE 12. To the left, the parallel arcs a_1 and a_2 and their images $\phi(a_1)$ and $\phi(a_2)$, and the discs Δ and $\phi(\Delta)$ these pairs cut from S . To the right, potential edges of R (shaded) along a_1 are swapped for edges along a_2 .

Proof. Labeling the arcs such that $\mathbf{y} = \{y_1, y_2, \dots, y_m, y_{m+1}, \dots, y_n\}$, $\mathbf{x} = \{x_1, x_2, \dots, x_m, y_{m+1}, \dots, y_n\}$, where $x_i \neq y_i$, we see that each of $n_{\mathbf{x}}(\mathcal{D})$ and $n_{\mathbf{y}}(\mathcal{D})$ are at least $m/4$, $|\mathbf{x}| = n$, and $|\mathbf{y}| \geq n - m + 1$. Now, referring to Section 2.2, as $J_+ = 0$, we have

$$m \leq 2(n_{\mathbf{y}}(\mathcal{D}) + n_{\mathbf{x}}(\mathcal{D})) = 1 + |\mathbf{x}| - |\mathbf{y}| = 1 + n - |\mathbf{y}| \leq 1 + n + m - n - 1 = m,$$

promoting each inequality to an equality. We conclude that each of $n_{\mathbf{y}}(\mathcal{D})$ and $n_{\mathbf{x}}(\mathcal{D})$ realize its lower bound of $m/4$, which implies that all corners of \mathcal{D} are acute and isolated, and hence, by the Gauss-Bonnet theorem, $e(\mathcal{D}) = \chi(\mathcal{D}) - m/2$. Finally, as $1 = \mu(A) = n_{\mathbf{y}}(\mathcal{D}) + n_{\mathbf{x}}(\mathcal{D}) + e(\mathcal{D})$, we have $\chi(\mathcal{D}) = 1$, from which the conclusion is immediate. \square

We will also make use of the following observation:

Lemma 5.4. *Let R be a level 0, positive region with isolated corners in (S, ϕ, \mathbf{a}) , and \tilde{R} a lift in the universal cover of S . Then for any other lift \tilde{R}' , $\partial\tilde{R} \cap \partial\tilde{R}'$ is contained in the interior of either $\partial\tilde{R}|_{\tilde{\mathbf{a}}}$ or $\partial\tilde{R}|_{\widetilde{\phi(\mathbf{a})}}$.*

Proof. Let p_0 be a point in $\partial\tilde{R} \cap \partial\tilde{R}'$. As the lifts are distinct, we may assume p_0 is in the interior of an edge of $\partial\tilde{R}$, along $\tilde{\mathbf{a}}$ (respectively, $\widetilde{\phi(\mathbf{a})}$). Now, as corners of R are isolated, p_0 is a transverse intersection point of the boundaries of the lifts, so there is a component ρ_0 of $\partial\tilde{R} \cap \partial\tilde{R}'$ connecting p_0 to another point p_1 , again in the interior of the same edge of $\partial\tilde{R}$. Then letting ρ_1 denote the component of $\partial\tilde{R} \cap \partial\tilde{R}'$ with endpoint p_1 , ρ_1 is along $\widetilde{\phi(\mathbf{a})}$ (respectively, $\tilde{\mathbf{a}}$), so has other endpoint p_2 in some other edge of $\partial\tilde{R}$, along $\tilde{\mathbf{a}}$ (respectively, $\phi(\mathbf{a})$). Repeating the argument until we return to p_1 , the conclusion is immediate. \square

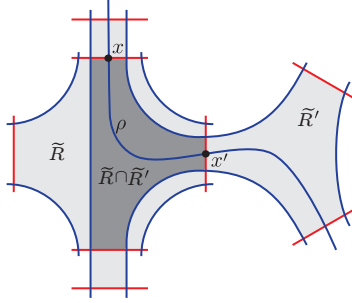


FIGURE 13. A possible intersection of two lifts \tilde{R} and \tilde{R}' of a region R . Note that $\partial\tilde{R} \cap \partial\tilde{R}' \subset \text{int}(\partial\tilde{R}|_{\tilde{\mathbf{a}}})$, while $\partial\tilde{R}' \cap \partial\tilde{R} \subset \text{int}(\partial\tilde{R}'|_{\widetilde{\phi(\mathbf{a})}})$

Our main use of Lemma 5.4 will be as follows: given a region $R \in \mathcal{R}_0^+(S, \phi, \mathbf{a})$ with isolated corners, for any point $x \in \partial R|_{\mathbf{a}} \cap \phi(\mathbf{a})$ (respectively, $\partial R|_{\phi(\mathbf{a})} \cap \mathbf{a}$), there is a unique point $x' \in \partial R|_{\mathbf{a}} \cap \phi(\mathbf{a})$ (respectively, $\partial R|_{\phi(\mathbf{a})} \cap \mathbf{a}$) such that x and x' are the endpoints of an arc ρ contained in an element of $\phi(\mathbf{a})$ (respectively, \mathbf{a}), and a lift of ρ to the universal cover of S is a properly embedded arc in a lift of R . It follows directly from Lemma 5.4 that x is in the interior of R if and only if the same is true of x' (Figure 13).

Recall that, given a collection of pairwise disjoint properly embedded arcs \mathbf{a} in a surface S , a *sum* of arcs a_1 and a_2 with endpoints $x_i \in \partial a_i$ adjacent on a common boundary component of S refers to the isotopy class of arc gotten by isotoping x_1 and x_2 to coincide and then taking the union of the pair of arcs at that point.

Lemma 5.5. *Let (S, ϕ) be an open book decomposition, \mathbf{a} an arc collection which contains a basis, and let $\mathbf{a}' := \mathbf{a} \cup \{a\}$, where a is a sum of two elements of \mathbf{a} . Then $\mathcal{R}_0^+(S, \phi, \mathbf{a}) = \emptyset$ only if $\mathcal{R}_0^+(S, \phi, \mathbf{a}') = \emptyset$.*

Proof. Let $\mathcal{R}_0^+(S, \phi, \mathbf{a}) = \emptyset$, and suppose $\mathcal{R}_0^+(S, \phi, \mathbf{a}') \neq \emptyset$, so there is some $R \in \mathcal{R}_0^+(S, \phi, \mathbf{a}')$ with $n > 0$ edges along a . We will show then that there is an arc collection \mathbf{a}''

obtained from \mathbf{a}' by adding arcs parallel to those in \mathbf{a} , and $R' \in \mathcal{R}_0^+(S, \phi, \mathbf{a}'')$ with $m < n$ edges along a . In particular then there is a one with no edges along a , and so an element of $\mathcal{R}_0^+(S, \phi, \mathbf{a})$, a contradiction.

Now, as \mathbf{a} contains a basis, it cuts S into discs, one of which contains a . By adding parallel copies of the bounding arcs of this disc, we obtain an arc collection \mathbf{a}_0 such that each element is parallel to an element of \mathbf{a} , R has no edge along any element of \mathbf{a}_0 , and a component of $S \setminus \mathbf{a}_0$ is a disc S_0 containing a and no element of \mathbf{a} . We label the boundary arcs a_0, a_1, \dots, a_{n-1} counter-clockwise along ∂S_0 . We then, for each i , let σ_i denote the unique component of $\phi(a_i) \cap S_0$ with endpoint in ∂a_i (Figure 14a). Observe that each σ_i must exit S_0 through a_{i+1} (taking indices mod n), else σ_i is an edge of some element of $\mathcal{R}_0^+(S, \phi, \mathbf{a})$ (Figure 14b).

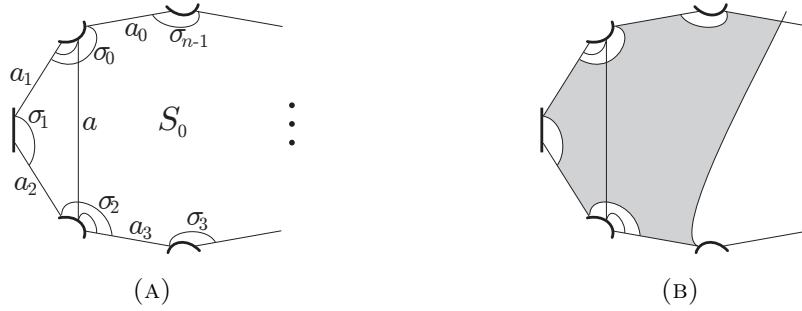


FIGURE 14

Now let Δ denote the disc component of $S \setminus \{a_1, a_2, a\}$ in S_0 . To simplify the notation, we orient each of the a_i such that Δ lies to its left following the orientation, give a the orientation induced as a sum of the a_i , and label ∂a such that the orientation goes from x^- to x^+ (Figure 15a).

We will then consider the case that x^+ is a corner of R ; the argument for x^- follows immediately by switching the roles of arcs and their images throughout the proof.

Consider then an edge of R along a , with endpoints x^+ and a point y in the interior of a . Let σ_y denote the unique component of $\phi(a) \cap \Delta$ with endpoint y . As $\mathcal{R}_0^+(S, \phi, \mathbf{a}) = \emptyset$, either σ_y exits Δ through a_1 or a_2 , or $\sigma_y = \sigma_2$; we consider the three cases separately:

- (1) For the case that σ_y exits Δ through a_2 , we may simply modify R by cutting along $\phi(a_2)$, and extending the result over a 4-gon in Δ , swapping the edge along a for one along a_2 (Figure 15b) (note that this directly generalizes the “shifting” employed in an elementary collapse). The result is then a boundary-based region R' , which we claim is again a level-0 positive region, as desired. To see this, note that we may assume after adding parallel arcs that each corner of R is isolated, and further that those corners of R' in the interior of S are either corners of R (so isolated), or one of the two new corners $y' \in a_2$ and $z' \in \phi(a_2)$. It follows from Lemma 5.4 (in particular the discussion it precedes) that y' is isolated if and only if y is isolated. For z' , let z denote the point $\sigma_2 \cap a$, so z and z' are endpoints of an arc in R along $\phi(a_2)$. Now, z is adjacent to R_z , so not in the interior of R , and thus again by Lemma 5.4, z' is isolated, as desired.

- (2) For the case that σ_y exits Δ through a_1 , we again modify R by cutting along $\phi(a_2)$, now extending the result over a 6-gon in Δ , swapping the edge along a for three edges along a_1, σ_1 , and a_2 (Figure 15c). That the resulting R' is a level-0 positive region follows directly from the argument of the previous case.

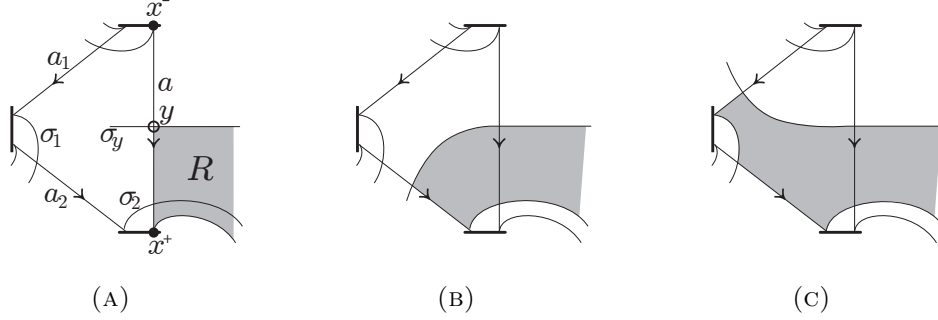


FIGURE 15

- (3) The remaining case, that $\sigma_y = \sigma_2$, is a straightforward generalization of the previous cases, but requires a little more effort to describe. Observe firstly that, as $y \in a \cap \phi(a_2)$, R has an edge along a_2 with endpoints x_2^- and a point in the interior of a_2 which we label y_2 . We then let ρ denote the component of $\phi(\mathbf{a}_0) \cap S_0$ with endpoint y_2 (Figure 16a).

Observe that ρ cannot be σ_1 , else R is exactly $\phi(\Delta) \setminus R_z$ and so fails condition (4) in Definition 5.1. It follows that ρ must exit S_0 through some a_i . We may then cut R along $\phi(a_1)$, extending the result, which no longer has the edge along a , over a_2 into S_0 , thus swapping the edge along a_2 for one along a subcollection of the \mathbf{a}_0 and $\phi(\mathbf{a}_0)$ (Figure 16b). Again, the verification that R' is a level-0 positive region follows as in case (1).



FIGURE 16. To the left, the pair of brackets is meant to signify an identification; i.e. what goes in one comes out the other.

□

Combining the above with Theorem 4.9, we have:

Corollary 5.6. *Let (M, ξ) be a closed contact 3-manifold, and (S, ϕ) be an open book decomposition of M supporting ξ . Suppose that there exists a pairwise disjoint properly embedded collection of arcs \mathbf{a} on S which contains a basis and is such that $\mathcal{R}_0^+(S, \phi, \mathbf{a}) = \emptyset$. Then $\mathfrak{o}(M, \xi) > 0$.*

It is worth pointing out that, combined with Theorem 4.9, the above corollary gives an independent proof of the result of [Wan15a], that $\mathcal{R}_0^+(S, \phi, \mathbf{a}) = \emptyset$, for \mathbf{a} a basis, implies tightness of the supported contact structure.

6. A FAMILY OF EXAMPLES

In this section, we present an infinite family of contact 3-manifolds $\{(M_{k,m}, \xi_{k,m})\}$, where $k \geq 2$ and $m > k$, such that $\mathfrak{o}(M_{k,m}, \xi_{k,m}) \leq k$ and, for $k > 2$, $0 < \mathfrak{o}(M_{k,m}, \xi_{k,m})$. Our examples are motivated by [CKK14] and are supported by the open book decompositions $(S, \phi_{k,m})$, where S is planar with four boundary components, and $\phi_{k,m}$ is a composition of k positive Dehn twists about each boundary component, and m negative twists about a curve which separates S into two pairs-of-pants (Figure 17). It follows from consideration of fractional Dehn twist coefficients, as shown by [IK15, Corollary 1.2] (cf. [Wan15a, Theorem 3.10]), that $\xi_{k,m}$ is tight. On the other hand, the ‘cap-off cobordism’ argument of [Bal13, Corollary 1.3] implies that $(M_{k,m}, \xi_{k,m})$ has vanishing Ozsváth–Szabó contact class.

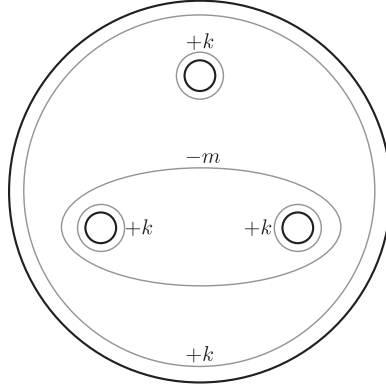


FIGURE 17

Part 1. This part gives bounds on \mathfrak{o} for the contact 3-manifolds $(M_{k,m}, \xi_{k,m})$.

Proposition 6.1. $\mathfrak{o}(M_{k,m}, \xi_{k,m}) \leq k$.

Proof. As noted above, the Ozsváth–Szabó contact class is zero. Hence, having fixed a basis of arcs \mathbf{a} on S , there must exist some $\mathbf{y} = \{y_1, y_2, y_3\}$ and a holomorphic domain \mathcal{D} in the corresponding Heegaard diagram (Σ, β, α) such that $\partial(\partial\mathcal{D}|\alpha) = \mathbf{y} - \mathbf{x}_\xi$, where as usual $\mathbf{x}_\xi = \{x_1, x_2, x_3\}$. Our goal is to show, using the basis of arcs depicted in Figure 18, that there is a unique (up to an obvious symmetry) such \mathcal{D} , and use this to conclude that $\mathfrak{o}(S, \phi, \mathbf{a}) = k$, from which the claim follows immediately.

We begin with a couple of observations to which we will make frequent reference throughout the argument:

- (1) At most one endpoint of each a_i may be a corner of \mathcal{D} , and furthermore $y_i = x_i$ if and only if neither endpoint of a_i is a corner of \mathcal{D} .
- (2) Consider a small disc about some $v \in a_i \cap \phi(a_j)$, modeled on the origin in the standard xy -plane, with a_i and $\phi(a_j)$ the y and x axes respectively, and label the multiplicity of \mathcal{D} in the i^{th} quadrant by q_i (Figure 19). Then either v is a corner of \mathcal{D} , in which case $q_1 + q_3 = q_2 + q_4 - 1$, or it is not a corner of \mathcal{D} and $q_1 + q_3 = q_2 + q_4$. In particular, note that whenever $q_2 = 0 = q_4$, the point v is not a corner of \mathcal{D} , and $q_1 = 0 = q_3$.

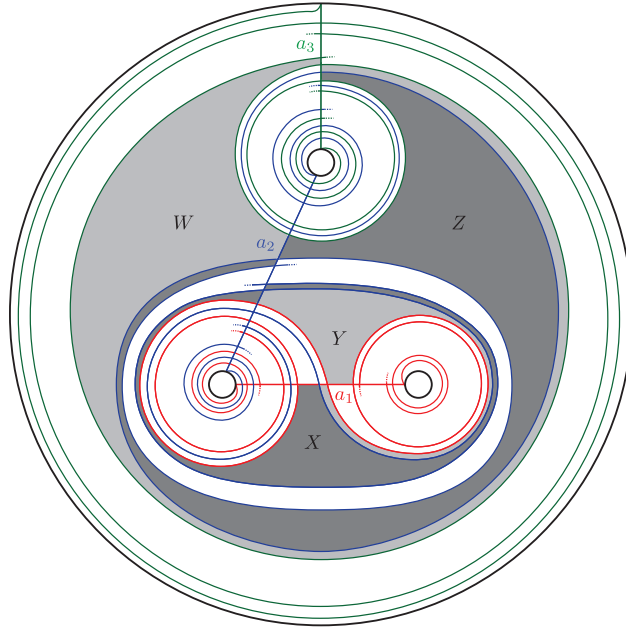


FIGURE 18. A basis of arcs $\mathbf{a} = \{a_1, a_2, a_3\}$ on S and the effect of the monodromy $\phi_{k,m}$ on that basis. Notice that, unlike elsewhere in the paper, we are using color to indicate *indices*, so that each pair of an arc and its image receives a unique color. The four hexagonal components of $S \setminus (\mathbf{a} \cup \phi(\mathbf{a}))$ are shaded and labeled.

Now consider the basis of arcs $\mathbf{a} = \{a_1, a_2, a_3\}$ depicted in Figure 18. We will use the symbol $|\cdot|$ to denote the multiplicity of a given region in \mathcal{D} . Observe that $S \setminus (\mathbf{a} \cup \phi(\mathbf{a}))$ consists of six 3-gon components of R_z (one for each point of $\partial\mathbf{a}$), some number of 4-gons into which the annular neighborhoods of the curves about which the Dehn twists are defined are cut, and finally four 6-gon components (which we label W, X, Y, Z as in Figure 18). As will become evident, we will need to keep track of only those 4-gons which are bounded by $\{a_1, a_2, \phi(a_1), \phi(a_2)\}$; to accomplish this, we label:

- those bounded by $a_1 \cup \phi(a_1)$ by A_i , $0 < 1 < k$, such that A_1 has a corner in ∂a_1 , and for $1 < i < k - 1$, ∂A_i shares two edges with each of A_{i-1} and A_{i+1} ,

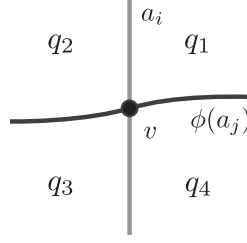


FIGURE 19

- those with an edge in each of $a_1, a_2, \phi(a_1)$, and $\phi(a_2)$ by B_i , $0 < i < 2k - 1$, or C_j , $0 < j < 2k$, where B_1 has a corner in ∂a_1 , C_1 has a corner in ∂a_2 , and each $B_i \cap B_{i+1}$ and $C_j \cap C_{j+1}$ is contained in $\phi(\mathbf{a})$, and
- a subset of those bounded by $a_2 \cup \phi(a_2)$ by D_i , $0 < i < k - 1$, where D_{k-2} shares 2 edges with X , and for $1 < i < k - 2$, ∂D_i shares two edges with each of D_{i-1} and D_{i+1} .

(See Figure 20).

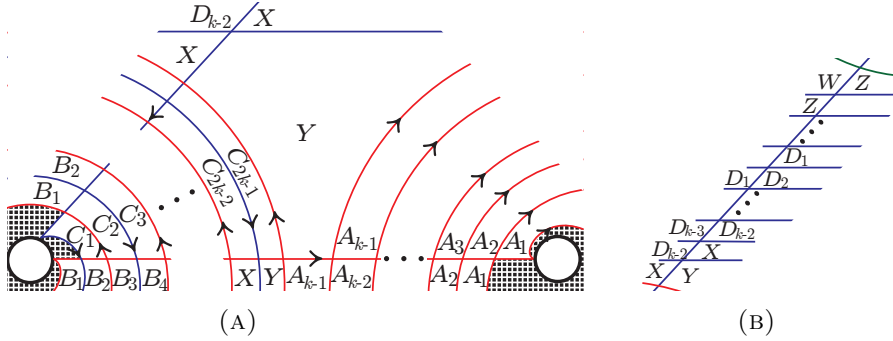


FIGURE 20. To the left, the regions along a_1 . To the right, those along a_2 . The shading indicates the region with basepoint, R_z , and the arcs have been oriented to make the argument more easily followed.

To begin, we will show that $|Y| \neq 0$ if and only if $|A_1| = |C_1| = 1$. To see this, note that by considering points along a_1 , and applying our observation (2) repeatedly, it follows that if $|Y| = 0$ then each $|A_i| = 0$, while conversely if $|A_1| = 0$ then $|A_i| = 0$ for each i , and hence $|Y| = 0$. Therefore we have $|Y| \neq 0$ if and only if $|A_1| = 1$. On the other hand, if $|Y| \neq 0$, and hence $|A_1| = 1$, then $|B_1| = 0$ from (1). Now suppose $|C_1| = 0$. Applying (2) to the points at the intersection of B_1 , B_2 , and C_1 , and of B_1 , C_1 , and C_2 , we deduce respectively that $|B_2| = 0$ and $|C_2| = 0$. Repeating this argument $2k - 2$ times, we find that $|C_i| = 0$, $|B_i| = 0$ for each i , and $|X| = 0$. Applying (2) to the point at the intersection of X , Y , C_{2k-1} , and C_{2k-2} , we are thus in contradiction with the assumption that $|Y| \neq 0$.

Because of the symmetry that allows us to interchange a_1 and a_3 , similar arguments apply to the region W and the region at the other endpoint of a_2 . Hence, by (1), $|Y|$ and $|W|$ cannot both be non-zero. Now, as $a_1 \cap \phi(a_3) = \emptyset = a_3 \cap \phi(a_1)$, and $a_i \cup \phi(a_i)$

bounds no bigons for any i , it must be that $x_2 \neq y_2$; using the obvious symmetry in the configuration (i.e. switching a_1 and a_3), we conclude that exactly one of Y and W has non-zero multiplicity. We may then assume without loss of generality that $|W| = 0$, which implies that $|A_1| = |C_1| = 1$. Note that since along a_2 only $\phi(a_2)$ separates Y from W , the segment of $\partial\mathcal{D}$ along $\phi(a_2)$ must start at x_2 and continue outward past Y ; if \mathcal{D} were to turn a corner earlier, the rest of $\phi(a_2)$ would be missed by $\partial\mathcal{D}$, connecting Y and W in the interior of \mathcal{D} and hence forcing them to have the same multiplicity. We conclude that y_2 lies on a_2 between its intersections with Y and W , so $y_i \in a_i \cap \phi(a_i)$ for each i .

In particular, no corner of any C_i along a_2 is a corner of \mathcal{D} . Considering points along a_2 and applying (2), we conclude that (setting $|B_0| = |C_0| = 0$), $|B_i| + |C_i| = |B_{i-1}| + |C_{i+1}|$ for each i . Similarly, considering points along a_1 , we find that $|B_i| + |C_i| = |B_{i+1}| + |C_{i-1}|$ for each i unless the point of intersection of B_i, C_i, B_{i+1} and C_{i-1} is the corner y_1 , in which case $|B_i| + |C_i| + 1 = |B_{i+1}| + |C_{i-1}|$.

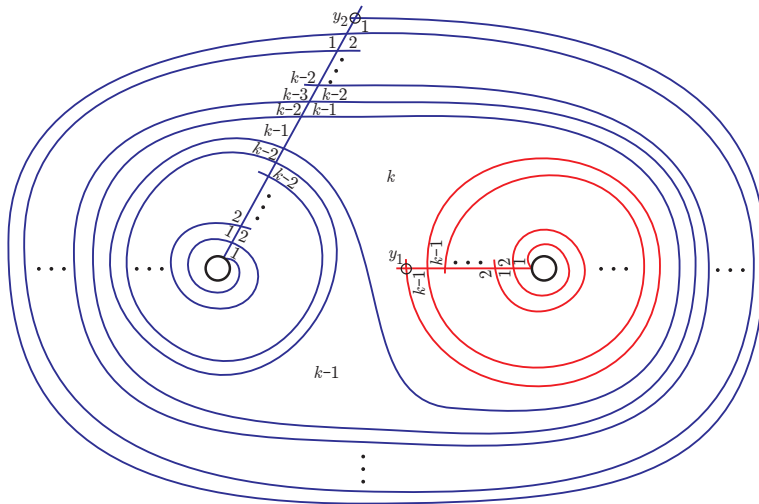
Suppose then that y_1 is a corner of C_{2j} (equivalently, of C_{2j-1} , for $j < k$). As $C_1 = 1$, and $B_1 = 0$, we find that, for $i \leq j$, $|B_{2i-1}| + 1 = |B_{2i}| = |C_{2i-1}| = |C_{2i}| = i$, while for $i > j$, we have $|B_{2i}| = |C_{2i}| = j$, $|B_{2i-1}| = j - 1$, and $|C_{2i-1}| = j + 1$. Again considering points along a_1 , we then have $|X| = j - 1$, $|Y| = j$, and so $|A_{k-1}| = j - 1 < k - 1$. However, as in this case no corner of any A_i is a corner of \mathcal{D} , and $A_1 = 1$, we find that $|A_i| = i$ for each i , a contradiction. We conclude then that $|C_{2k-1}| = k = |Y|$, so $|A_{k-1}| \geq k - 1$. On the other hand, considering multiplicities around corners of the A_i , we see that $|A_{k-1}| \leq k - 1$, so $|A_{k-1}| = k - 1$, and $C_{2k-1} \cap A_{k-1}$ is our corner y_1 . It is immediate then that this determines all remaining multiplicities of labeled regions, with $|A_i| = |D_i| = i$ (Figure 21).

On the other hand, let \mathcal{D}' denote a domain representing a class in $\hat{\pi}_2(\vec{y}, \vec{x}')$, for some \vec{x}' . As no domain may leave x_3 , we find that $y_3 = x_3 = x'_3$. Considering multiplicities in \mathcal{D}' , it is then immediate that $|D_1| = 1$. Having fixed the corners in y , all inequalities in the above argument are now promoted to equalities; this determines all remaining multiplicities as above, so $\mathcal{D} = \mathcal{D}'$, and $\vec{x}' = \vec{x}_\xi$. Assuming then that \mathcal{D} has a holomorphic representative, $\hat{\partial}_{HF}\vec{y} = \vec{x}_\xi$, as desired.

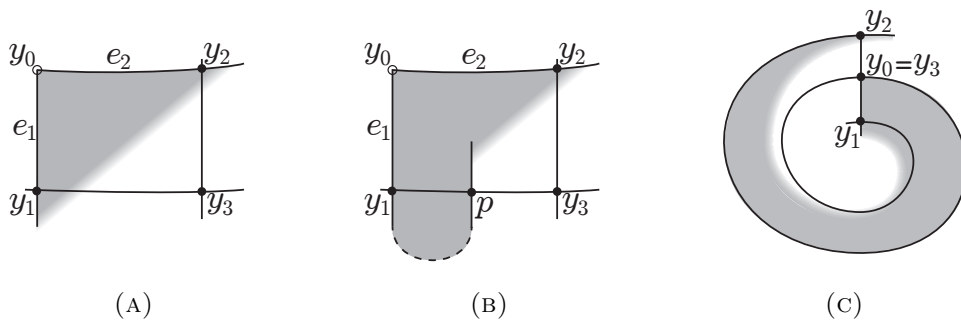
Finally, again exploiting the obvious symmetry which allows us to interchange a_1 and a_2 , the case $x_3 \neq y_3$ is identical; switching indices throughout we find a further candidate domain \mathcal{D}'' . As these are the only possible candidates, we may assume \mathcal{D} has a holomorphic representative. We have $n_{\vec{y}}(\mathcal{D}) = (4k - 1)/4 + 1/4 = k$, $n_{\vec{x}_\xi}(\mathcal{D}) = 1/4 + 1/4 = 2/4$, and $|\vec{y}| = |\vec{x}_\xi| = 3$, so $J_+(\mathcal{D}) = 2k$ by (2.6). We conclude $o(S, \phi, \mathbf{a}) = k$, and hence $\mathbf{o}(M, \xi) \leq k$. \square

Having proved an upper bound on \mathbf{o} for each member of the family contact 3-manifolds $(M_{k,m}, \xi_{k,m})$, we now move on to prove that there exists a non-zero lower bound on \mathbf{o} for this family using the method developed in Section 5. In doing so, we will make use of a simple observation concerning boundary-based regions:

Lemma 6.2. *Suppose that $R \in \mathcal{R}(S, \phi, \mathbf{a})$ contains some corner y_0 , so there are incident edges of ∂R which we label e_1 along \mathbf{a} and e_2 along $\phi(\mathbf{a})$. Suppose further that there are points $y_1, y_2, y_3 \in \mathbf{a} \cap \phi(\mathbf{a})$ such that $y_i \in \text{int}(e_i)$ for $i = 1, 2$, and the four y_i are acute*



corners of a 4-gon in $\mathbf{a} \cup \phi(\mathbf{a})$, embedded in its interior, with those edges incident to y_0 contained in the edges e_i . Then y_3 is in the interior of R .



Proof. The setup is illustrated in Figure 22a. Suppose that y_3 is not in the interior of R , and let D denote the 4-gon with corners the four y_i . Fixing a lift \tilde{R} in the universal cover of S , there is then a path in $\partial\tilde{R}$ comprised of a lift each of e_1 and e_2 , joined at a lift of y_0 . Letting \tilde{y}_1 denote the lift of y_1 on this path, there is a connected component $\tilde{\sigma}$ of $\phi(\mathbf{a}) \cap \tilde{R}$ with endpoint \tilde{y}_1 . As y_3 is assumed not in the interior of R , it has no lift in the interior of $\tilde{\sigma}$; therefore, the image of $\tilde{\sigma}$ in S is an arc contained in the edge of ∂D from y_1 to y_3 . We denote its other endpoint by $p \in \partial D$. Let \tilde{R}_0 denote the component of $\tilde{R} \setminus \tilde{\sigma}$ whose boundary does not contain our fixed lift of y_0 . The image of $\partial\tilde{R}_0$ on S is thus a regular-null-homotopic closed curve which follows along ∂R from y_1 to p , then along ∂D back to y_1 (Figure 22b). On the other hand, following along ∂R into D from p , as our path ∂R cannot have two

consecutive corners in D (since corners of R alternate between point on the interior and the boundary of S), and has no obtuse corners, it must (after eliminating any bigons it might form with $[y_1, y_3]$) exit through either e_1 after turning no corners or e_2 after turning at a point in the interior of D . However, such a path in either case cannot be the boundary of an immersed disc, giving a contradiction. \square

Our main usage of Lemma 6.2 will be to exclude certain points of intersection of arcs from appearing as corners in level-0 positive regions. As a particular application which will be used repeatedly, suppose the 4-gon in the statement of the lemma is situated so that the corners y_0 and y_3 are identified (Figure 22c). It follows then that this intersection point must lie in the interior of any boundary-based region of which it is a corner, and so cannot be the corner of any level-0 positive region.

Proposition 6.3. $0 < \mathbf{o}(M_{k,m}, \xi_{k,m})$, for $k > 2$.

Proof. We wish to apply Corollary 5.6; as such, we will verify that $\mathcal{R}_0^+(S, \phi_{3,m}, \mathbf{a}) = \emptyset$ for some $m \gg 3$ (observe that the remaining cases are all obtained by Legendrian surgeries on these, so follow by Theorem 1.2). Suppose then that $R_0 \in \mathcal{R}_0^+(S, \phi_{3,m}, \mathbf{a})$.

As much of the following will require us to keep track of points in $\{a_1, a_2\} \cap \{\phi(a_1), \phi(a_2)\}$, we will let v_0 (respectively, w_0) denote the endpoint of a_1 (respectively, a_2) on the component of ∂S containing endpoints of both a_1 and a_2 , and v_i (respectively, w_i) the i^{th} intersection point along the interior of a_1 (respectively, a_2), increasing away from v_0 (respectively, w_0). Figure 23a illustrates several vertices labelled according to this convention. It follows immediately from Lemma 6.2, and in particular the observation which it precedes (Figure 22c), that no element of $\{v_3, v_4, v_7, w_3\}$ can be a corner of R_0 .

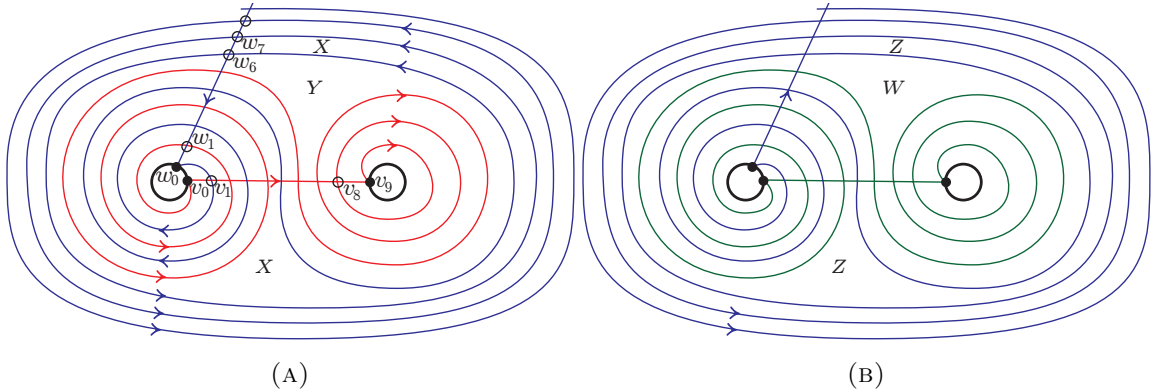


FIGURE 23. The two symmetric halves of the open book decomposition supporting $(M_{k,m}, \xi_{k,m})$. To the left we have a_1 and half of a_2 and the images of these; to the right the remaining half of a_2 along with a_3 , and their images.

Now, retaining the labeling of the four 6-gon regions of $S \setminus (\mathbf{a} \cup \phi(\mathbf{a}))$ as in Proposition 6.1 (Figure 23), we will make a couple of preliminary observations. Firstly, we will show

(now using $|\cdot|$ to denote multiplicities of R_0) that $|Y| > 0$ only if $|X| > 0$. Note that, by the symmetry of the configuration (i.e. swapping a_1 and a_3) this is equivalent to the statement that $|W| > 0$ only if $|Z| > 0$. Suppose then that $|Y| > 0$. As in Proposition 6.1, we may consider multiplicities about v_7 and v_8 to conclude that each of the 4-gons incident to these vertices has non-zero multiplicity, and conclude that v_9 is a corner. There is then an edge along a_1 from v_9 to some corner v_k , where as noted above $k \neq 7$. Now, if $k < 7$, then the interior of the segment of a_2 between w_5 and w_6 is in the interior of R ; as this segment is along the boundary of X , $|X| > 0$. Similarly, if $k = 8$, the segment of $\phi(a_1)$ from v_8 to v_0 is an edge of R_0 , again giving $|X| > 0$, as desired.

Next, observe that (as in Proposition 6.1) $a_1 \cap \phi(a_3) = \emptyset = a_3 \cap \phi(a_1)$, and $a_i \cup \phi(a_i)$ bounds no bigons for any i , so we must have an edge of ∂R_0 along a_2 , and hence may without loss of generality use the symmetry of the configuration to assume that w_0 is a corner. We will show then that this implies that $|X| > 0$: indeed, if w_0 is a corner, there is an edge along a_2 from w_0 to some w_l , where $l \neq 3$. Now, if $l > 6$, then this edge contributes to the boundary of X , giving $|X| > 0$. Similarly, if $l \in \{2, 4\}$ then the incident edge along $\phi(a_2)$ continues to contribute to the boundary of X , while if $l = 5$, then the interior of the segment of a_1 from v_1 to v_6 is in the interior of R , in each case giving $|X| > 0$. Finally, if $l \in \{1, 6\}$, then the edge along $\phi(a_1)$ from w_l contributes to the boundary of Y , which then has positive multiplicity, and so by the previous paragraph again $|X| > 0$, as desired.

Continuing with our assumption that w_0 is a corner, and hence $|X| > 0$, suppose then that $|Y| = 0$. Now, an edge along $\phi(a_1)$ with one corner at v_9 and the other at any of w_1, w_3 , or w_5 , or an edge along a_2 with one corner at w_0 and the other at some $w_{j>5}$, would contribute to the boundary of Y , giving a contradiction. Therefore, any edge along a_2 with a corner at w_0 also has a corner at w_2 or w_4 . However, each of these would imply an edge of ∂R_0 from w_j along $\phi(a_2)$ to ∂S . In particular then as each such segment contributes to the boundary of W , we have $|W| > 0$, so also $|Z| > 0$. In summary, either w_0 is a corner and each of $|X|$ and $|Y|$ are non-zero, or the other endpoint of a_2 is a corner, and each of $|W|$ and $|Z|$ are non-zero. Hence, we may again use the symmetry to assume without loss of generality that each of $|X|$ and $|Y|$ are non-zero, and w_0 is a corner. On the other hand, condition (4) of Definition 5.1 implies that any element of $\mathcal{R}_0^+(S, \phi, \mathbf{a})$ must miss at least one of the 6-gons, so $|W| = 0$. Finally then, supposing $|Z| \neq 0$, it is immediate that by increasing m by one (Figure 24), we find a region R in $\mathcal{R}_0^+(S, \phi, a)$ such that $|W| = |Z| = 0$, $|X|$ and $|Y|$ non-zero, and w_0 is the only corner in ∂a_2 . As such a region cannot have boundary in a_3 , we may restrict attention to $\mathcal{R}_0^+(S, \phi, \{a_1, a_2\})$.

We wish now to analyze potential corners of R . Note that as only one endpoint of a_2 appears as a corner, any corner in a_2 co-bounds an edge along a_2 with w_0 . On the other hand, a corner at any point v_i in the interior a_1 can potentially co-bound an edge along a_1 with either v_0 or v_9 ; we use the notation v_i^- for the former, v_i^+ for the latter situation (Figure 25a). Also, as any path in ∂R is determined by the list of corners at which it turns, we will often make use of such lists as shorthand for paths.

As noted above, Lemma 6.2 has eliminated the possibility of a corner at v_3, v_4, v_7 , and w_3 . On the other hand, as any corner c along the interior of $\phi(a_2)$ must co-bound an edge along $\phi(a_2)$ with w_0 , we can travel along this edge from w_0 , then must turn left at

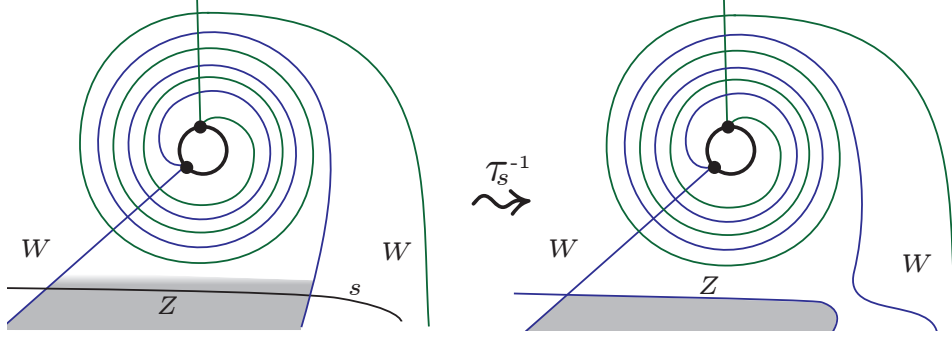


FIGURE 24

c ; however, such a turn at either of w_2 or w_4 sends us to the other endpoint of a_2 , which is known not to be a corner, while such a turn at v_1 or v_5 corresponds to the corners v_1^+ or v_5^+ . Therefore, none of w_2, w_4, v_1^- , and v_5^- can be corners. Consider next the point w_5 : again appealing to Lemma 6.2, with w_0, w_3, w_2 and w_5 in place of (respectively) y_0, y_1, y_2 , and y_3 in the statement of the lemma, we observe that if R has an edge along a_2 from w_0 to w_5 , then each of w_2 and w_3 must be interior to an edge of R incident to w_0 , implying that w_5 is in the interior of R , a contradiction (Figure 25b). To eliminate v_6 as a corner, we first consider v_6^+ . Once again appealing to Lemma 6.2, now with v_9, v_7, v_8 and w_5 in place of (respectively) y_0, y_1, y_2 , and y_3 , we conclude that to avoid overlapping v_6 we have a path $v_6^+ v_9 v_8$, which then must continue along a_1 to v_0 (Figure 25c). Now with v_0, v_4, v_2 and v_6 in place of (respectively) y_0, y_1, y_2 , and y_3 , to avoid overlap with v_6 the continuation of our path from v_0 along $\phi(a_1)$ must then turn at v_2 ; repeating the argument we see that interior corners henceforth alternate between v_2 and v_8 , and in particular the path never returns to v_6 , a contradiction. An identical argument rules out v_6^- . As a final application of Lemma 6.2, consider v_5 , which as noted above can only appear as the corner v_5^+ . Then with v_5, v_6, v_3 and v_4 in place of (respectively) y_0, y_1, y_2 , and y_3 (Figure 25d), we see that v_4 is in the interior of R . But then either v_5 is also in the interior of R , or $\phi(a_2)$ appears with both possible orientations in ∂R ; either case gives a contradiction. Our potential corner set is thus $\{v_0, v_1^+, v_2, v_8, v_9, w_0, w_1, w_{k \geq 6}\}$ (Figure 26a).

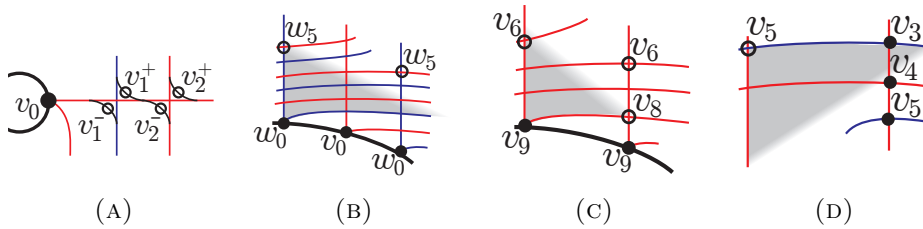


FIGURE 25. Figure (a) illustrates our corner-labeling convention. Figures (b), (c) and (d) illustrate (for simplicity, in the universal cover of S) the 4-gons used in applying Lemma 6.2 to eliminate vertices.

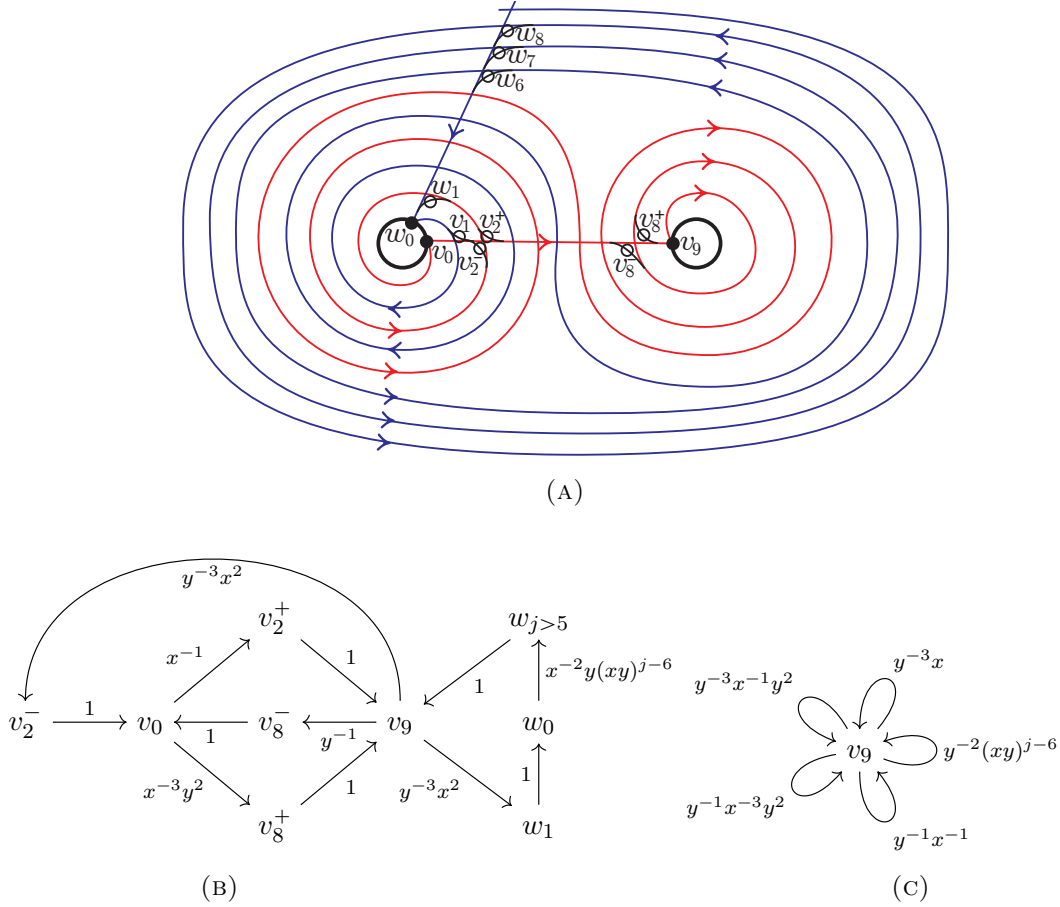


FIGURE 26

Now, as v_9 is a corner, ∂R (with the standard counter-clockwise orientation) determines an oriented loop in $\mathbf{a} \cup \phi(\mathbf{a})$ which contains v_9 , turns only at our potential corners, and then always to the left, and alternates between corners on ∂S and those in the interior. Each such loop then determines an element of $\pi_1(S, v_9)$. In fact, as clearly we may restrict attention to the subgroup generated by the simple loops x and y about the boundary components containing (respectively) v_1 and v_9 , we obtain an element in the free group generated by x and y . As ∂R is null-homotopic, its boundary must represent the identity. As a further restriction, consider an occurrence of w_1 as a corner, which clearly can appear only in the path $v_9 w_1 w_0$. Observe that, as R is an immersed disc, this path cannot then turn at v_1 , and therefore must continue to some $w_{j>5}$.

We will use this to show that v_2 cannot be a corner. First, note that if ∂R contains any corner $w_{j>5}$, it contains the path $w_j w_0$, and moreover, using Lemma 6.2, if this path does not then turn at v_1 , then v_2 is in the interior of R (Figure 26a). Supposing then that v_2 is a corner, it follows that each such path turns at v_1 . Considering all possible paths from v_9 which satisfy these criteria, and the corresponding words in x and y (Figure 26c), we

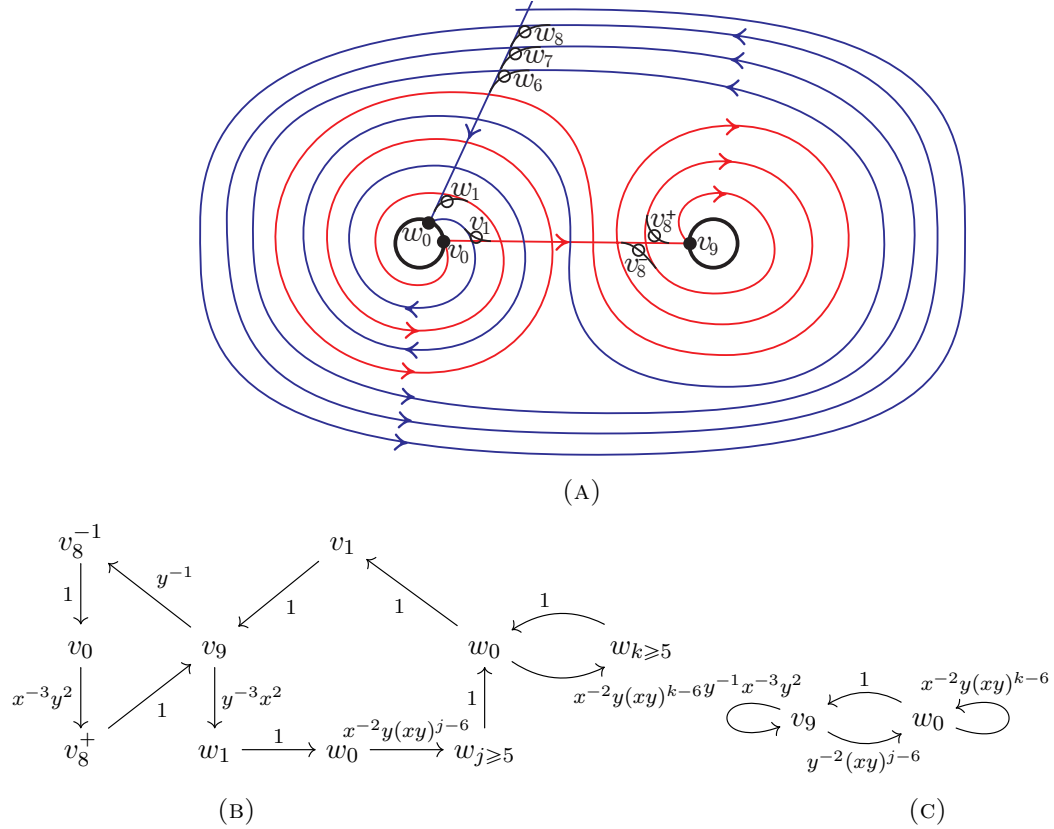


FIGURE 27

find that ∂R is a concatenation of the following 5 paths which begin and end at v_9 (and corresponding words in x and y):

- (1) $v_9 w_1 w_0 w_j v_9 \rightsquigarrow y^{-2}(xy)^{j-6}$ (for $j \geq 6$)
- (2) $v_9 v_8^- v_0 v_2^+ v_9 \rightsquigarrow y^{-1}x^{-1}$
- (3) $v_9 v_8^- v_0 v_8^+ v_9 \rightsquigarrow y^{-1}x^{-3}y^2$
- (4) $v_9 v_2^- v_0 v_8^+ v_9 \rightsquigarrow y^{-3}x^{-1}y^2$
- (5) $v_9 v_2^- v_0 v_2^+ v_9 \rightsquigarrow y^{-3}x$

However, we observe that each of the corresponding words ends in either $y^k(xy)^l$ for some $k < 0$ and $l \geq 0$, x^{-1} , $x^{-1}y^2$, or x , and moreover that this property is preserved under concatenation with words (1)-(5) and resulting cancellation. As such, no concatenation gives the identity, so v_2 cannot be a corner of R .

Our potential corner set has thus been reduced to $\{v_0, v_1^+, v_8^\pm, v_9, w_0, w_1, w_{k \geq 6}\}$ (Figure 27a). Again studying the space of loops at v_9 (Figure 27c), we find that any loop corresponds to a reduced word in x and y which ends in either xy^p for some $p > 0$, $x^{-1}y^q$ for some $q \geq 0$, or y^{-2} . \square

Part 2. To better illustrate how the spectral sequence is used in computations, we choose the monodromy of the open book decomposition $(S, \phi_{2,3})$ in its isotopy class so that the effect of the monodromy on the basis of arcs \mathbf{a} is depicted in Figure 28. Note that $\widehat{\partial}_{HF}\vec{\mathbf{v}} = \vec{\mathbf{x}}_\xi + \vec{\mathbf{u}}$ via the domains $\mathcal{D}_1 = A + B_1 + 2B_5 + 2C + D + F + G + H$ and $\mathcal{D}_2 = B_3$ and $\widehat{\partial}_{HF}\vec{\mathbf{y}} = \vec{\mathbf{u}}$ via the domain $\mathcal{D}_3 = B_1 + B_2$ where $\mathbf{y} := \{y_1, y_2, x_3\}$, $\mathbf{u} := \{u_1, v_2, x_3\}$, and $\mathbf{v} := \{v_1, v_2, x_3\}$. We leave the verification of these claims to the reader. Hence, $\widehat{\partial}_{HF}(\vec{\mathbf{y}} + \vec{\mathbf{v}}) = \vec{\mathbf{x}}_\xi$.

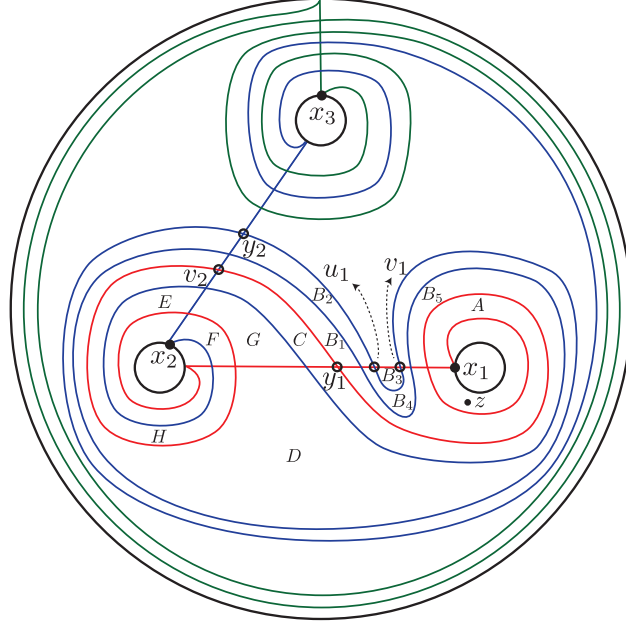


FIGURE 28

Next we move on to compute the J_+ values for each of the domains $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$. In this regard, first note that

$$|\mathbf{v}| = 2, \quad |\mathbf{u}| = 2, \quad |\mathbf{y}| = 3.$$

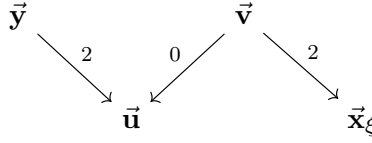
As for point measures,

- $n_{\mathbf{v}}(\mathcal{D}_1) = \frac{6}{4}$ and $n_{\mathbf{x}}(\mathcal{D}_1) = \frac{2}{4}$,
- $n_{\mathbf{v}}(\mathcal{D}_2) = \frac{1}{4}$ and $n_{\mathbf{u}}(\mathcal{D}_2) = \frac{1}{4}$,
- $n_{\mathbf{y}}(\mathcal{D}_3) = \frac{2}{4}$ and $n_{\mathbf{u}}(\mathcal{D}_3) = \frac{2}{4}$.

Using (2.6), we deduce that

$$\begin{aligned} J_+(\mathcal{D}_1) &= 2\left(\frac{6}{4} + \frac{2}{4}\right) - 1 + 2 - 3 = 2, \\ J_+(\mathcal{D}_2) &= 2\left(\frac{1}{4} + \frac{1}{4}\right) - 1 + 2 - 2 = 0, \\ J_+(\mathcal{D}_3) &= 2\left(\frac{2}{4} + \frac{2}{4}\right) - 1 + 3 - 2 = 2. \end{aligned}$$

We can describe the generators and pseudo-holomorphic curves involved in the demise of the Ozsváth–Szabó contact class via a weighted directed graph:



Focusing on this graph and using the description of the spectral sequence associated to (S, ϕ', \mathbf{a}) from Section 2.3, we compute an upper bound for $o(S, \phi', \mathbf{a})$. Recall that the generator \vec{x}_ξ is ∂_i -closed for any $i \geq 0$. Hence $\vec{x}_\xi \in Z^k(S, \phi', \mathbf{a}; J_{HF})$ for any $k \geq 1$. Meanwhile, $\vec{x}_\xi \neq \partial_0 b_0$, or $\vec{x}_\xi \neq \partial_0 b_0 + \partial_1 b_1$ with $\partial_0 b_1 = 0$ for any b_0, b_1 in the span of \vec{y} , \vec{u} , and \vec{v} . On the other hand, setting $b_2 = \vec{y}$, $b_1 = \vec{y} + \vec{v}$, and $b_0 = \vec{v}$, we have $\vec{x}_\xi = \partial_0 b_0 + \partial_1 b_1 + \partial_2 b_2$, with $\partial_0 b_2 = 0$ and $\partial_1 b_2 + \partial_0 b_1 = 0$. Therefore, $\vec{x}_\xi \in \mathcal{B}^3(S, \phi', \mathbf{a}; J_{HF})$, and $o(S, \phi', \mathbf{a}) \leq 2$.

7. FINAL REMARKS

We end with a few natural questions that we will investigate in future work. First, we would like to know whether spectral order detects overtwistedness. In this regard, we ask the following question:

Question 7.1. *Does $\mathbf{o}(M, \xi) = 0$ imply that ξ is overtwisted?*

Note that an affirmative answer to Question 7.1 along with the non-decreasing behavior of \mathbf{o} under Legendrian surgery, Theorem 1.2, would provide an alternative proof of the following theorem, which has recently been proved by the last author in [Wan15b]:

Theorem 7.2. *Let ξ be a tight contact structure on M , and $K \subset M$ be a null-homologous Legendrian knot. Then, contact (-1) -surgery on K produces a 3-manifold with a tight contact structure.*

To be more explicit, if (M, ξ) is a closed contact 3-manifold where ξ is a tight contact structure, and (M', ξ') is obtained from (M, ξ) via Legendrian surgery, then $\mathbf{o}(M, \xi) > 0$ and $\mathbf{o}(M, \xi) \leq \mathbf{o}(M', \xi')$ (Theorem 1.2) would imply that $\mathbf{o}(M', \xi') > 0$, which in turn would imply that ξ' is tight by Theorem 4.1.

Another question of interest is related to generalizing our invariant to compact contact 3-manifolds with convex boundary. In this regard, Definition 2.1 readily generalizes to the case of partial open book decompositions introduced in [HKM09a]. This allows us to extend Definition 2.2 to compact contact 3-manifolds with convex boundary. This was independently observed by Juhász and Kang who used it to find an upper bound on the spectral order for a closed contact 3-manifold that contains a Giroux torsion domain [JK16]. With this understood, we would like to compare \mathbf{o} to Wendl’s *planar torsion* [Wen13]. As is stated in [LW11, Theorem 6], planar torsion provides an upper bound to Latschev and Wendl’s algebraic torsion. Moreover, planar torsion detects overtwistedness. One could expect a similar relationship between spectral order and Wendl’s planar torsion.

Question 7.3. *Suppose that the closed contact 3-manifold (M, ξ) has planar k -torsion. Does it imply $\mathbf{o}(M, \xi) \leq k$?*

Next, we would like to understand how \mathbf{o} behaves under exact symplectic cobordisms, generalizing the case of Stein cobordisms:

Question 7.4. *Is spectral order non-decreasing under exact symplectic cobordisms?*

Since spectral order is defined using open book decompositions, and there is currently no topological description of exact symplectic cobordisms in general, it may be hard to answer this question.

Finally, we would like to know more about possible values of \mathbf{o} in the case when the Ozsváth–Szabó contact class is non-zero:

Question 7.5. *Does there exist a closed contact 3-manifold (M, ξ) with non-vanishing Ozsváth–Szabó contact class and finite $\mathbf{o}(M, \xi)$?*

An affirmative answer to this question would give a way to distinguish symplectically fillable contact 3-manifolds from Stein fillable ones.

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